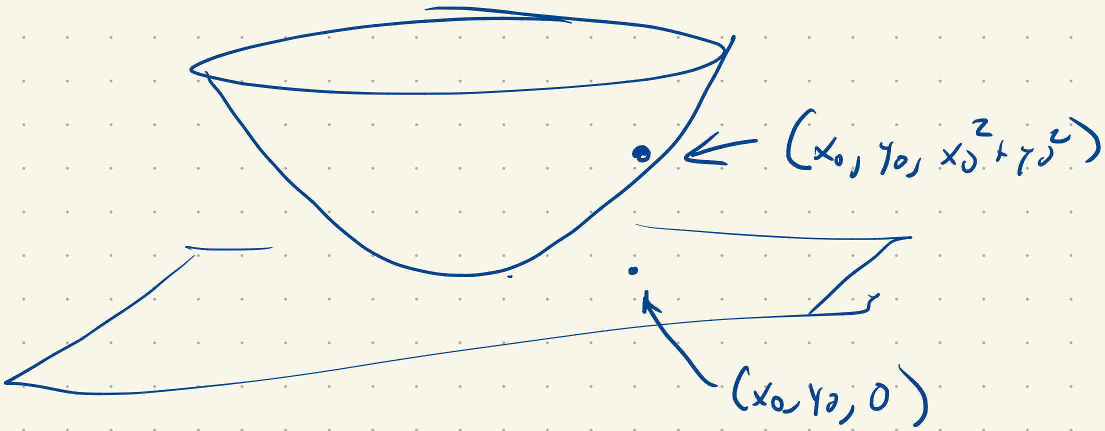
Normal lines + tangent planes

Suppose $z = f(x,y)$. e.g. $z = x^2 + y^2$



You spent a lot of time in calc I thinking about tangent lines to a surface. The right analog here is the tangent plane

For concreteness $x = 1, y = 2 \Rightarrow z = 5$

I'd like to find the tangent plane at this point.

To describe a plane we need 2 pieces of info.

1) a point on the plane

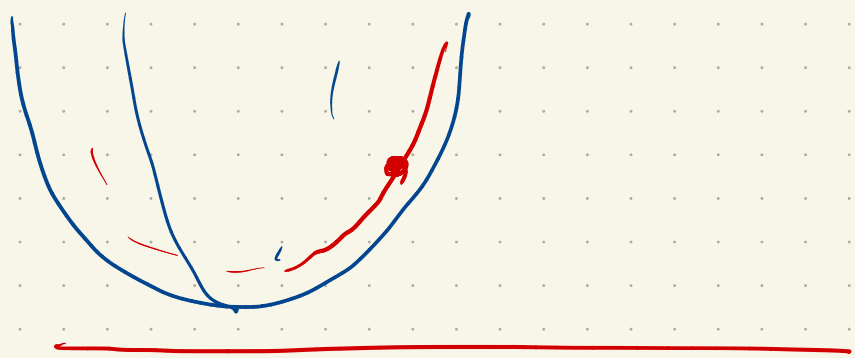
2) a normal vector.

So here, we have the point: $(1, 2, 5)$.

We need a normal vector.

t^2+4

Consider $\vec{r}(t) = \langle t, 2, f(t,2) \rangle$



This is a curve entirely in the surface.

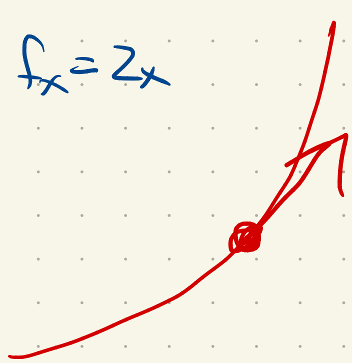
The tangent to this curve is always tangent to the surface.

$$\vec{r}\,'(t) = \langle 1, 0, f_x(t,2) \rangle$$

$$\vec{r}\,'(t) = \langle 1, 0, f_x(1,2) \rangle$$

$f_x = 2x$

$$= \langle 1, 0, 2 \rangle$$



We can play this same game to find another vector tangent at the same point

$$\vec{s}(t) = \langle 1, t, f(1,t) \rangle$$

$$\vec{s}\,'(t) = \langle 0, 1, f_y(1,t) \rangle \qquad t = 2$$

$$\vec{s}\,'(t) = \langle 0, 1, f_y(1,2) \rangle$$

$$= \langle 0, 1, 4 \rangle$$

So now I have two vectors tangent to the surface at this point

$$\langle 1, 0, f_x(1,2) \rangle = \langle 1, 0, 2 \rangle$$

$$\langle 0, 1, f_y(1,2) \rangle = \langle 0, 1, 4 \rangle$$

In fact if $\vec{v} = (a,b)$ another is

$$\langle a, b, a f_x + b f_y \rangle = \langle a, b, \vec{\nabla} f \cdot \vec{v} \rangle$$

$$\langle 1+at, 2+bt, f(1+at, 2+bt) \rangle$$

Let me be more terse:

$$\langle 1, 0, f_x \rangle$$
$$\langle 0, 1, f_y \rangle.$$

Can we make a normal direction?

$$\begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{matrix} \qquad \langle -f_x, -f_y, 1 \rangle$$

Traditionally we use $\langle f_x, f_y, -1 \rangle$. It's just as $\perp$.

So: at $(1, 2, f(1,2))$ $\quad f_x = 2x = 2$

$f_y = 2y = 4$

$\langle 2, 4, -1 \rangle$ is normal

$$2(x-1) + 4(y-2) - (z-5) = 0$$

$$z = 5 + 2(x-1) + 4(y-2)$$

$$z = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$$

Look like anything? This is just the linear approximation!

$$z = L(x,y) \quad !$$

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e.g. $\quad xy + yz + zx = 11 \quad$ at $\quad (1,2,3)$

$$2 + 6 + 3$$

$$z(x+y) = 11 - xy$$

$$z = \frac{11 - xy}{x+y}$$

$$\frac{\partial z}{\partial x} = \frac{-y(x+y) - (11-xy)\cdot 1}{(x+y)^2}$$

$$= \frac{-2\cdot 3 - (11-2)}{9} = \frac{-6-9}{9} = \frac{-15}{9} = -\frac{5}{3}$$

$$\frac{\partial z}{\partial y} = \frac{-x(x+y) - (11-xy)\cdot 1}{9} = \frac{-1\cdot 3 - (11-2)\cdot 1}{9}$$

$$= \frac{-3-9}{9} = -\frac{12}{9} = -\frac{4}{3}$$

$$z = 3 - \frac{5}{3}(x-1) - \frac{4}{3}(y-2)$$

There is a better way!

$$F(x,y,z) = xy + yz + zx$$

level set of $F(x,y,z) = 11$ $\quad (1,2,3)$

$$\nabla F = \langle y+z,\ x+z,\ y+x \rangle$$

$$= \langle 5, 4, 3 \rangle$$

$$5(x-1) + 4(y-2) + 3(z-3) = 0$$

$$z = 3 - \frac{5}{3}(x-1) - \frac{4}{3}(y-2)$$ Whoa!

$\vec{r}(t)$ lives entirely in $F(x,y,z)=c$

$$F(\vec{r}(t)) = c$$

$$\frac{d}{dt} F(\vec{r}(t)) = 0$$

$$F_x \dot{x} + F_y \dot{y} + F_z \dot{z} = 0$$

$$\vec{\nabla} F \cdot \vec{r}\,' = 0$$

So we define the tangent plane to level set at (x_0, y_0, z_0) by

$$F_x(x - x_0) + F_y(y - y_0) + F_z(z - z_0) = 0$$