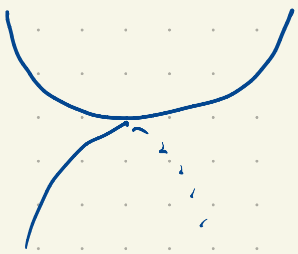
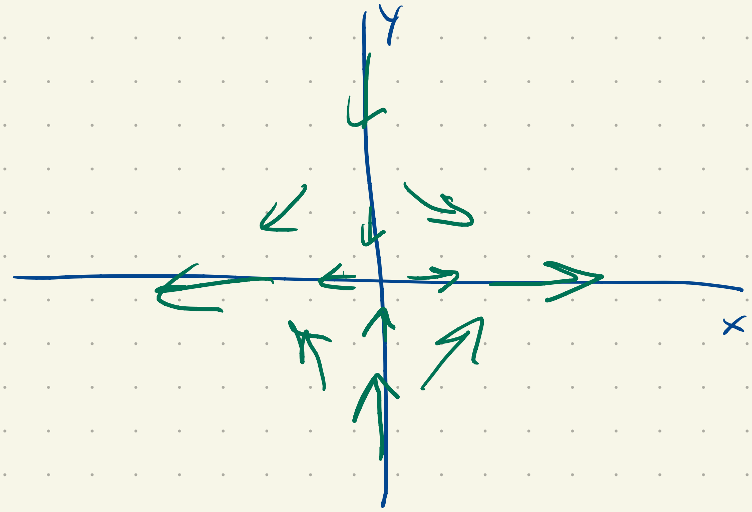
Last class: gradient of a function

new notation

$f(x,y)$ $\qquad \vec{\nabla} f = \langle \partial_x f, \partial_y f \rangle$ $\qquad f_y, \frac{\partial f}{\partial y}$

$f(x,y) = x^2 - y^2$





Point: If you are traveling with velocity $\vec{v}$,

then $\vec{\nabla} f \cdot \vec{v}$ tells you the rate of change of

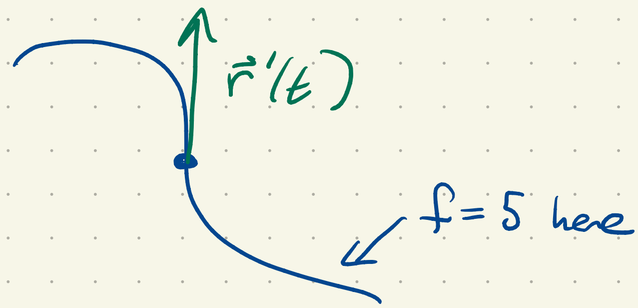
f as you travel.

Other claims:

1) $\vec{\nabla} f$ is perpendicular to level sets of f
2) $\vec{\nabla} f$ points in the direction of steepest increase of f.
3) $\|\vec{\nabla} f\|$ tells you about the steepness of the graph of f.

Let's demonstrate 1).

Consider a level set of f.



Consider a path $\vec{r}(t)$ entirely contained in the level set.

So $f(\vec{r}(t)) = 5$ for all t. So $\frac{d}{dt} f(\vec{r}(t)) = 0$.

But $\frac{d}{dt} f(\vec{r}(t)) = \nabla f \cdot \vec{r}\,'$

Hence $\vec{\nabla} f$ is perpendicular to every vector tangential to the level set.

Related concept: directional derivatives.

Suppose we have a temperature field $T(x,y)$.

By definition $D_{\vec{v}} T$ is the rate of change of

"directional derivative of T along $\vec{v}$"

T if you travel with velocity v. More formally,

$$D_{\vec{v}} T(\underset{(x_0, y_0)}{p}) = \left.\frac{d}{dt}\right|_{t=0} T(p + t\vec{v})$$

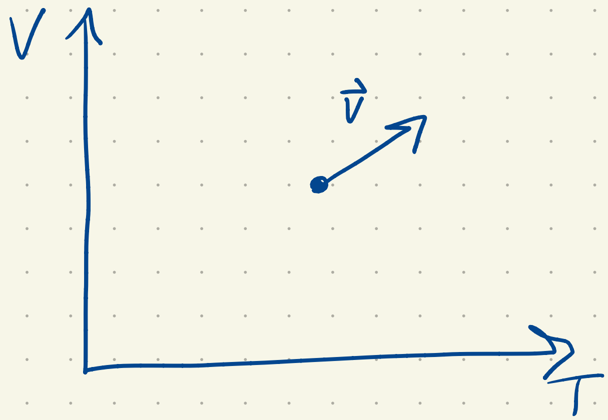
If you've been paying attention,

$$D_{\vec{v}}T = \vec{\nabla} T \cdot \vec{v}$$ (strictly speaking only when differentiable).

Your text only allows $\vec{v}$ to be a unit vector.

This is silly:

$$P = \frac{0.083\,T}{V}$$



It's natural to ask what is the rate of change of pressure as you move along $\vec{v}$. But what's a unit vector?

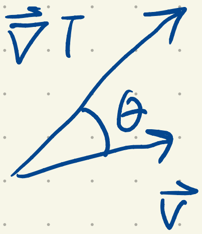
$1\,L = 1\,K$? It makes no sense.

But you should be aware of the text's convention.

But lets go back to $T(x,y)$.

Now $\quad D_{\vec{v}} T = \vec{\nabla} T \cdot \vec{v}$

$$= \|\vec{\nabla} T\| \, \|\vec{v}\| \cos\theta$$



If you double $\vec{v}$ you double $D_{\vec{v}} T$. Great. So let's stick with $\|\vec{v}\| = 1$ and ask which direction maximizes the rate of change.

$$\|\vec{\nabla} T\| \cos\theta$$

maximized when $\theta = 1$.

So $\vec{v}$ is parallel to $\vec{\nabla} T$. (Uphill!)

Moreover, $\|\vec{\nabla} T\|$ tells you $\dfrac{dT}{dt}$ when you travel in the increasing direction with "unit" speed.

We've been doing all this in 2d, but it works in 3d and more.

E.g. $T(x,y,z) = \dfrac{80}{1+x^2+2y^2+3z^2}$ C, temp dist.

x,y,z in cm

hot spot at origin, decays to 0.

$\vec{\nabla} T = \langle \partial_x T, \partial_y T, \partial_z T \rangle$

$$\frac{\partial T}{\partial x} = \frac{-80}{(1+x^2+2y^2+3z^2)^2} \cdot 2x = \frac{-160x}{(1+x^2+y^2+3z^2)^2}$$

$$\frac{\partial T}{\partial y} = \frac{-3200y}{(1+x^2+2y^2+3z^2)^2}$$

$3 \cdot 2 \cdot 80$

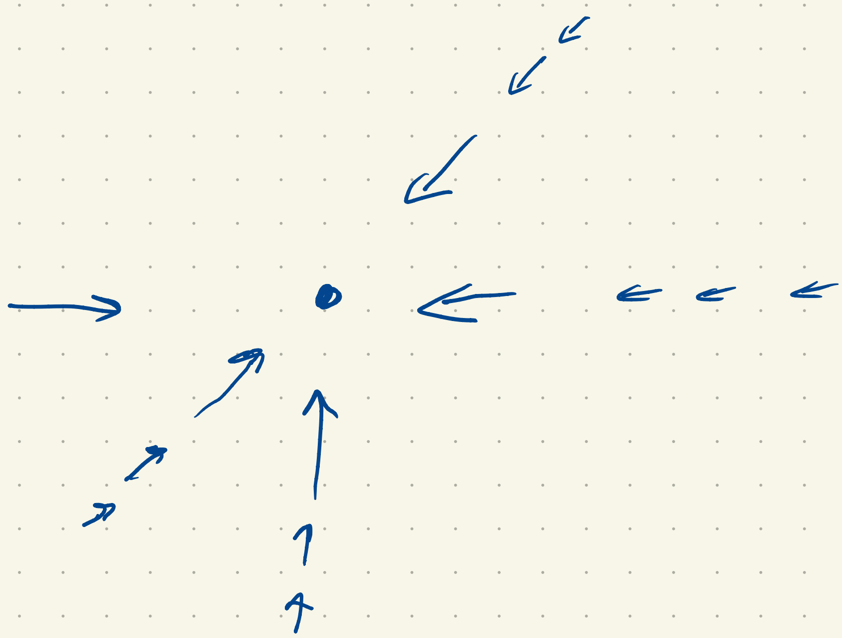
$$\frac{\partial T}{\partial z} = \frac{-480z}{(1+x^2+2y^2+3z^2)^2}$$

$$\vec{\nabla} T = \frac{-160}{(1+x^2+2y^2+3z^2)} \langle x, 2y, 3z \rangle \ {}^\circ C/cm$$

If you are at $(1,1,-2)$ cm, what direction has the steepest increase in temp? Express as a unit vector.

$\langle 1, 2, -6 \rangle$ $\qquad$ $1^2 + 2^2 + 6^2 = 1 + 4 + 36 = 41$

$$\vec{u} = \frac{1}{\sqrt{41}} \langle 1, 2, -6 \rangle$$

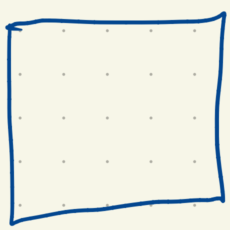


etc.

Trouble with boundaries:

min/max can happen on boundary.

If f is cts and domain is bounded and closed it attains a max/min:



$$f(x,y) = x^2 - 2xy + 2y$$

$$0 \leq x \leq 3$$
$$0 \leq y \leq 2$$

$$\frac{\partial f}{\partial x} = 2x - 2y \Rightarrow x = y$$

$$\frac{\partial f}{\partial y} = -2x + 2 \Rightarrow x = 1$$

$\Rightarrow 1, 1$

$$f(1,1) = 1 - 2 + 2 = 1$$

On $x = 0$ is $2y$ $\quad f(0,2) = 4$

On $x = 3$ is $9 - 6y + 2y = 9 - 4y$ $\quad y = 0,\ 9.$