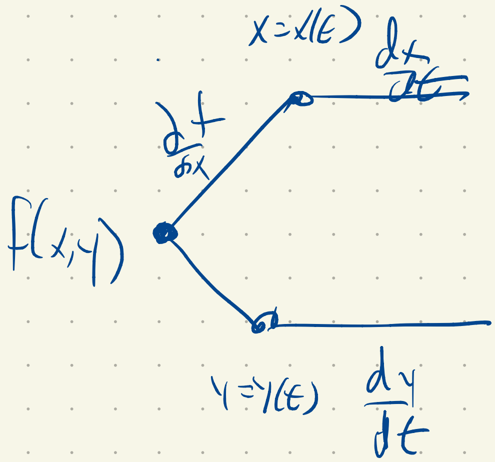
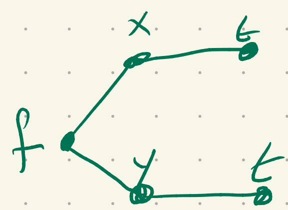
$$\frac{dy}{dt} = \sin\theta \frac{dr}{dt} + r\cos\theta \frac{d\theta}{dt}$$

Chain rule applies to each in term.

So we'll focus on the case of one output variable.



$$\frac{d}{dt} f = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$



What if f depends on x, y but

$$x = x(u,v)$$

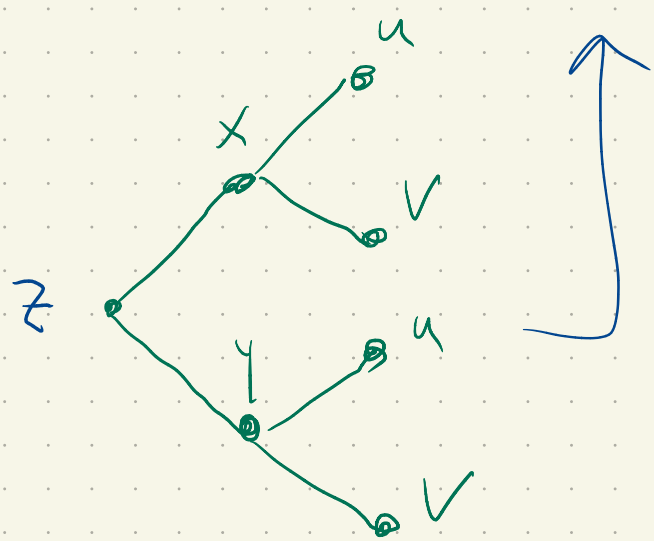
?

$$y = y(u,v)$$

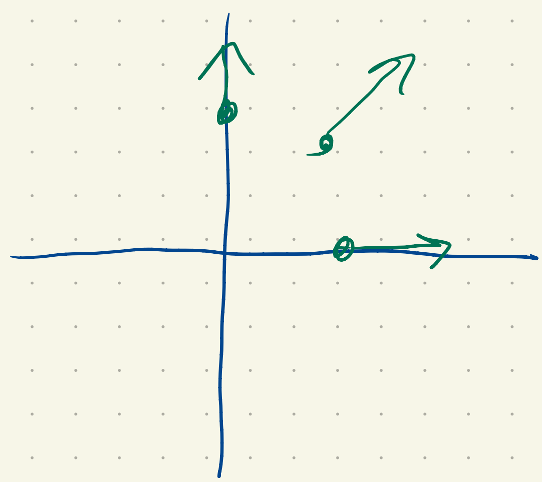
$$z = f(x(u,v), y(u,v))$$

$$\frac{\partial z}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$$



$$h(x,y) = xy$$



$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$\frac{\partial h}{\partial r} = \frac{\partial h}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial h}{\partial y}\frac{\partial y}{\partial r}$$

$$= y\cos\theta + x\sin\theta$$

$$= r\sin\theta\cos\theta + r\cos\theta\sin\theta$$

Directional Derivatives + the Gradient

$T(x,y)$ $\quad$ $\vec{r}(t) = \langle x(t), y(t) \rangle$

$$\frac{d}{dt} T(\vec{r}(t)) = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} \qquad \text{(chain rule)}$$

$$\vec{r}\,'(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle$$

$$w = \langle a, b \rangle$$

$$w \cdot \vec{r}\,'(t) = a\frac{dx}{dt} + b\frac{dy}{dt}$$

That is, $\displaystyle \frac{d}{dt} T(\vec{r}(t)) = \left\langle \frac{\partial T}{\partial x}, \frac{\partial T}{\partial y} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle$

We call $\displaystyle \left\langle \frac{\partial T}{\partial x}, \frac{\partial T}{\partial y} \right\rangle$ the gradient of T.

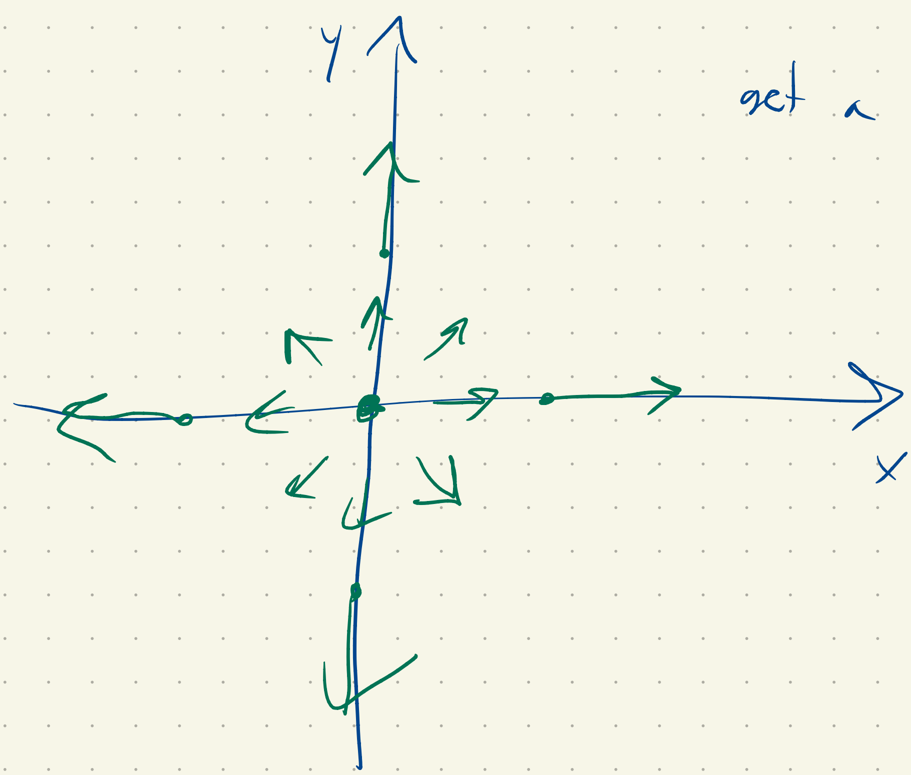
and write it as ∇T.

To get a better sense,

$$T(x,y) = x^2 + y^2$$

$$\nabla T = \langle 2x, 2y \rangle$$

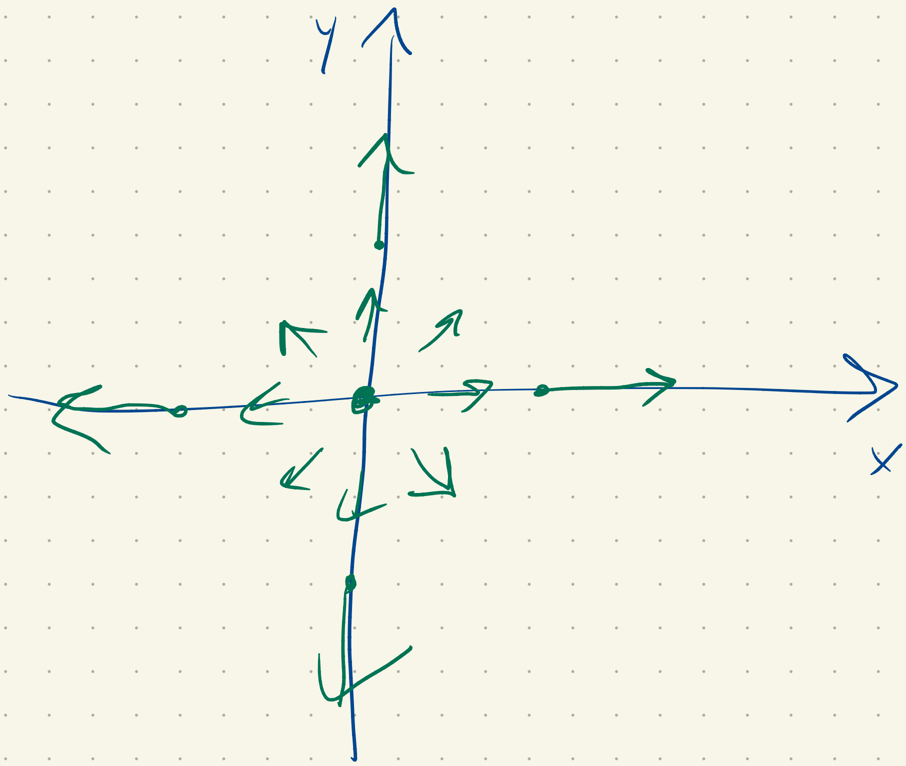
At position $\langle x,y \rangle$ get a vector ($\langle 2x, 2y \rangle$ in this case.



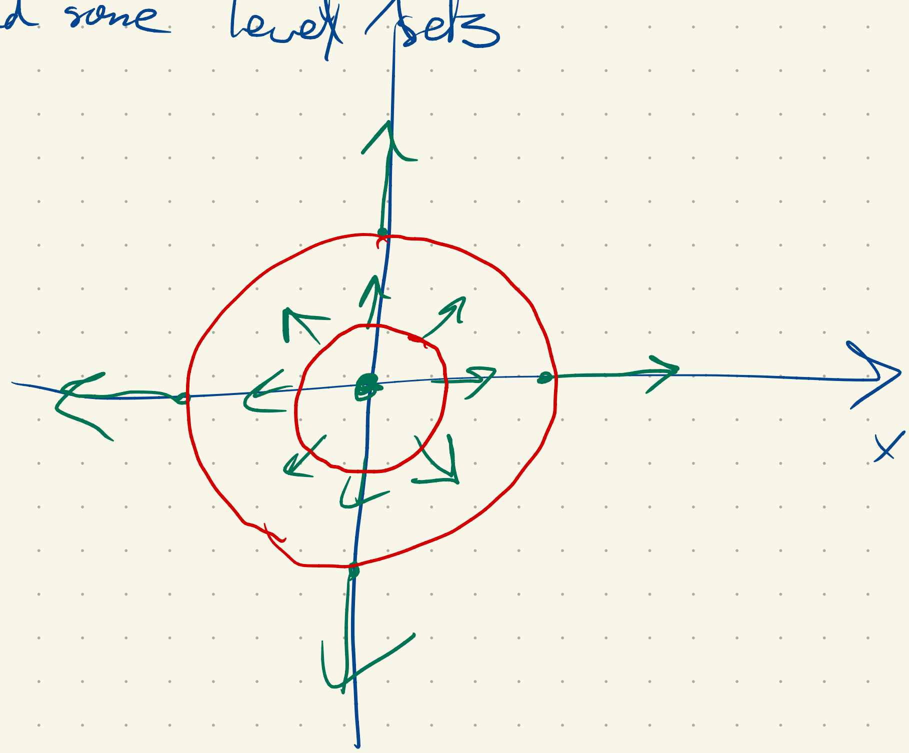
This is the job of the gradient:

If you are travelling with velocity $\vec{v}$,

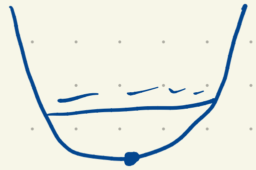
$\nabla T \cdot \vec{v}$ tells you the rate of change you see in T.



Let's add some level sets



And:



Some key points:

1) $\vec{\nabla} f = \langle f_x, f_y \rangle$ is a vector field

2) It points in the direction of steepest ascent ("up"!)

3) It is perpendicular to the level sets of f

4) It's length tells you about steepness of the graph

Note that $\nabla f = 0$ at 0, the flat part

5) Most important: for a curve $\vec{r}(t)$ in x-y space, $\nabla f \cdot \vec{r}\,'$ tells you about the rate of change of f along the curve.

---

5) Was how we introduced it.

Why 3)?

If f is a curve in a level set, $f \circ \vec{r}(t)$ is const.

So $\frac{d}{dt} f(\vec{r}(t)) = 0$

$\nabla f \cdot \vec{r}\,'(t)$

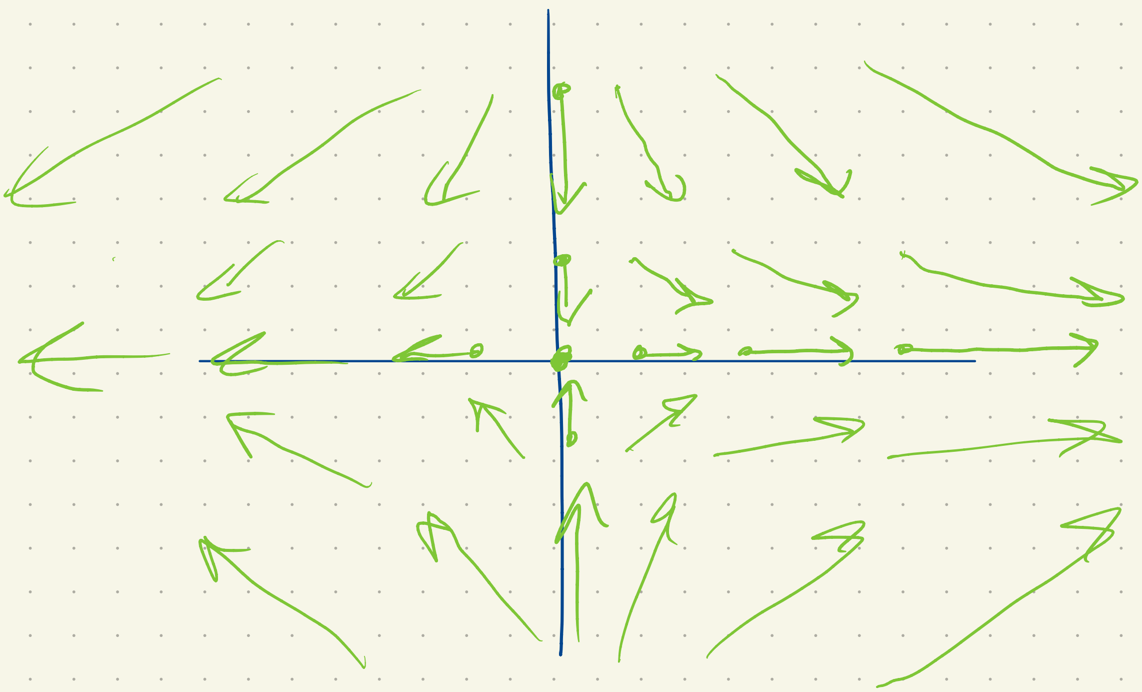
We'll come back to 2) shortly

Another example:

$$h(x,y) = x^2 - y^2 \qquad \text{(saddle)}$$

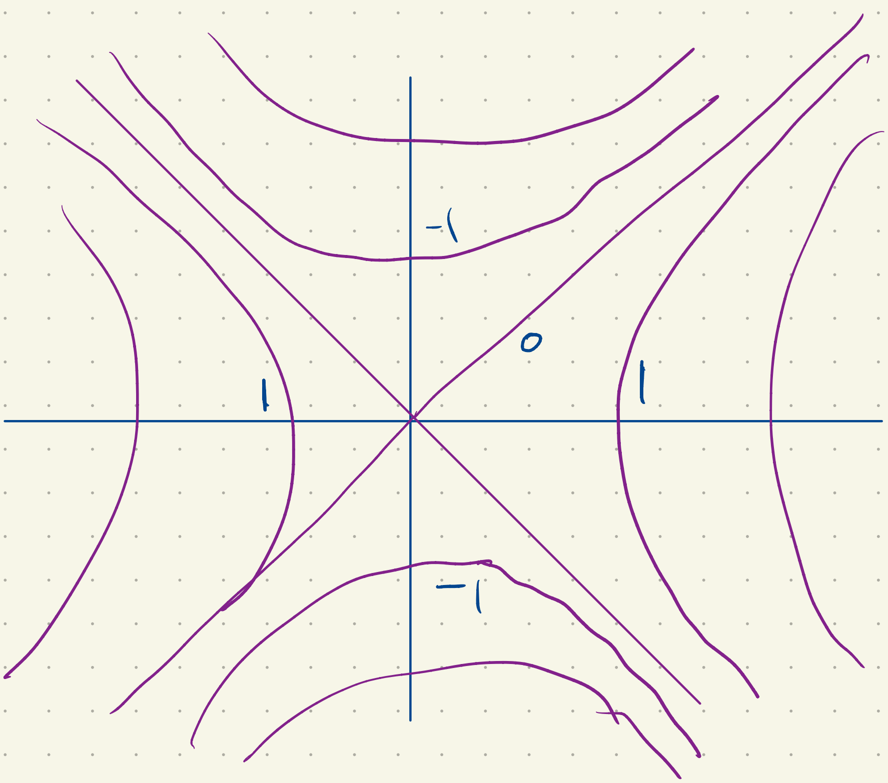
$$\vec{\nabla} h = \langle 2x, -2y \rangle$$

$\vec{\nabla} h$

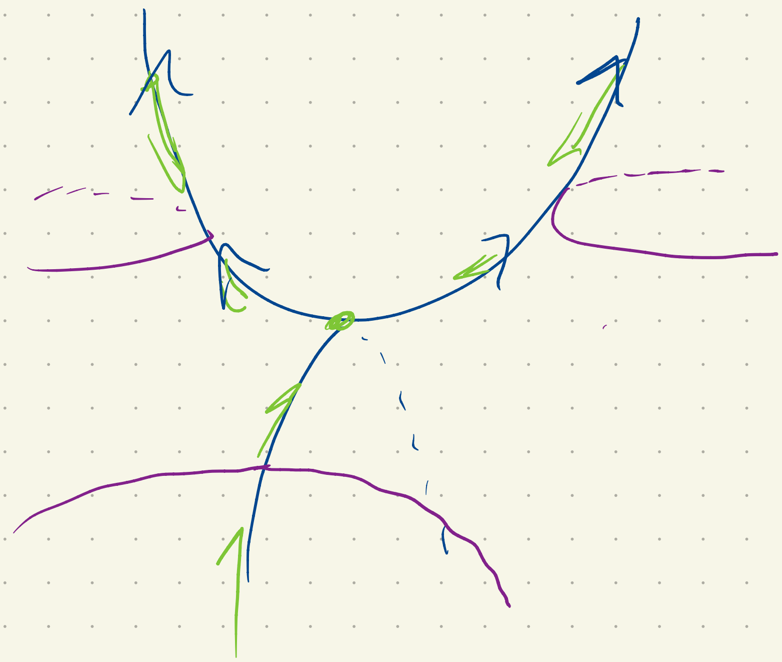


Note:

$\nabla h = 0$ at the saddle point.

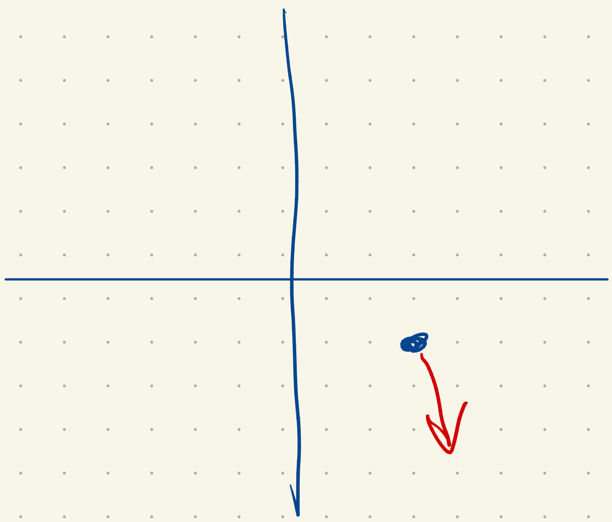


level sets



What is rate of change of h if travels

with velocity $\langle 1, -2 \rangle$ at $x = 3$ $y = -1$



$$\vec{\nabla} h = \langle 2x, -2y \rangle$$
$$= \langle 6, 2 \rangle \quad \text{(at 3,1!)}$$

$$\vec{\nabla} h \cdot \vec{v} = \langle 6, 2 \rangle \cdot \langle 1, -2 \rangle$$
$$= 6 - 4 = 2$$

Some justification:

uni

$$\vec{\nabla} f \cdot \vec{v} = \|\vec{\nabla} f\| \, \|\vec{v}\| \cos\theta$$

So if $\|\vec{v}\| = 1$, then $\vec{\nabla} f \cdot \vec{v}$ is biggest if $\cos\theta = 1$
$\theta = 0$.

And most negative if $\cos\theta = -1$, $\theta = \pi$.

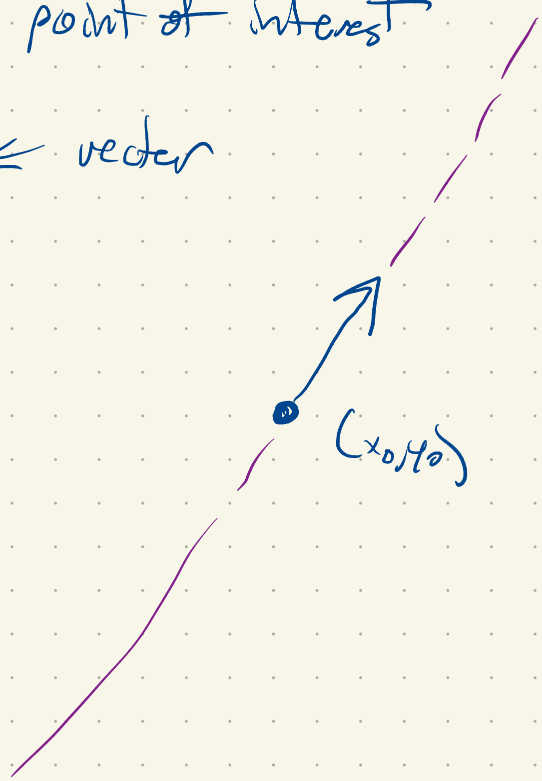
If $\cos\theta = \pi/2$, $\vec{\nabla} f \cdot \vec{v} = 0$, A related notion: directional derivatives.

(so related it'll be confusing at first)

$f(x,y)$

$(x_0,y_0) \leftarrow$ point of interest

$\vec{v} = \langle v_x, v_y \rangle \leftarrow$ vector

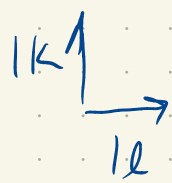


If I travel along this line, with the given velocity, what is the observed rate of change of f?

$$\frac{d}{dt}\Big|_{t=0} f(x_0+tv_x, y_0+tv_y) := D_{\vec{v}} f(x_0,y_0)$$

"Directional derivative of f at (x_0, y_0) along $\vec{v}$".

Note: your book only allows $\vec{v}$ to be a unit vector, which is silly. $P = 8.3\, T/V$



Now if you've been paying attention

$$D_{\vec{v}} f = \vec{\nabla} f \cdot \vec{v}$$

almost. There are functions which have directional derivatives in all directions but for which this formula is false.

$$f(x,y) = \frac{x^2 y}{x^2+y^2} \quad (0 \text{ at origin})$$

$(x_0, y_0) = (0,0)$

$v = \langle v_x, v_y \rangle$

$$f(tv_x, tv_y) = \frac{t^3 v_x^2 v_y}{t^2(v_x^2+v_y^2)}$$

$$= t\, \frac{v_x^2 v_y}{v_x^2+v_y^2}$$

$$\nabla_{\vec{v}} f = \frac{v_x^2 v_y}{v_x^2 + v_y^2}$$

But $f = 0$ on axes

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0.$$

$$\frac{x^2}{x^2+y^2} - \frac{2x^2y^2}{(x^2+y^2)^2} = \frac{x^4 - x^2y^2}{(x^2+y^2)^2}$$

along $y = x$ is 0

along $y = 2x$

is $\frac{x^4 - 4x^4}{4x^4}$

$-\frac{3}{4}$

Functions for which the tangent plane approx is good are called diff. For a diff function

$D_{\vec{v}} f = \vec{\nabla} f \cdot \vec{v}$. And

If $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and are ok

near (x_0, y_0) then f is diff at x_0, y_0.