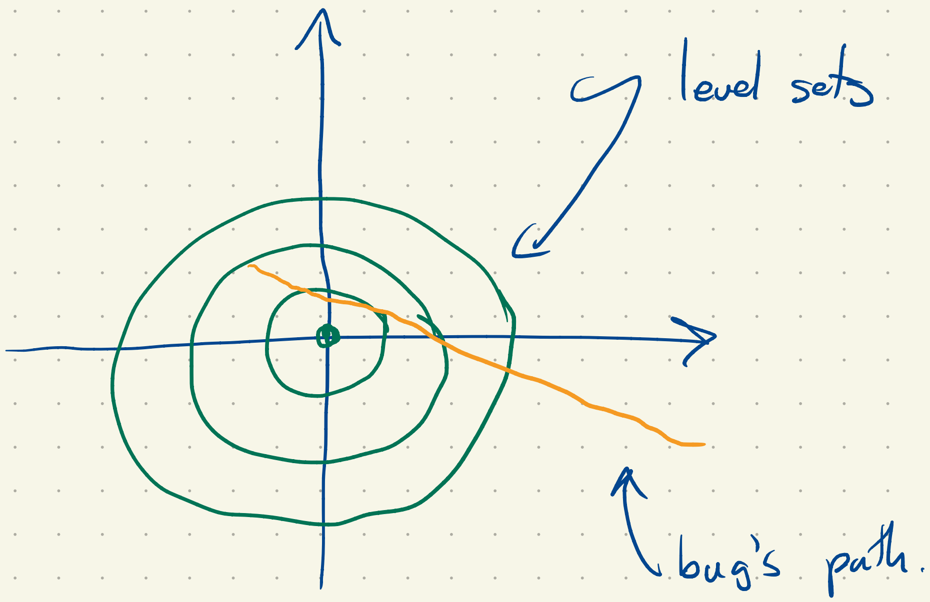
Chain Rule (a bug's tale.



$$T(x,y) = 100\, e^{-\left(\frac{x^2+y^2}{5}\right)} \quad {}^\circ C \qquad x, y \text{ in cm}$$

I have a bug walking around

$$\vec{r}(t) = \langle x(t), y(t) \rangle$$

$$= \langle 2+6t, 1-t \rangle \qquad t \text{ in s}$$

The bug has a thermometer and can keep track of the temperature $T(t) = T(x(t), y(t))$

Question: What is the rate of change of temp that the bug sees at $t=0$?

$$\frac{d}{dt}\Big|_{t=0} T(\vec{r}(t)) = ?$$

There's an inside and an outside function! It's the chain rule, but it's a little more complicated.

At $t=0$, $\vec{r}(0) = \langle 2, 1 \rangle$.

Let's replace $T(x,y)$ with it's linearization at $x=2, y=1$.

$$L(x,y) = T(x_0,y_0) + \frac{\partial T}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial T}{\partial y}(x_0,y_0)(y-y_0)$$

$$T(x_0,y_0) = 100e^{-1} \approx 36.7\,°C$$

$$\frac{\partial T}{\partial x} = 100\,e^{-\frac{x^2+y^2}{5}} \cdot \left(-\frac{2x}{5}\right)$$

$$= -40x\,e^{-\left(\frac{x^2+y^2}{5}\right)}$$

at $(x_0,y_0) = (2,1)$, $\frac{\partial T}{\partial x}(x_0,y_0) = -80e^{-1} = -29.4\ {}^\circ C/cm$

$$\frac{\partial T}{\partial y} = -40ye^{-\left(\frac{x^2+y^2}{5}\right)} \qquad \frac{\partial T}{\partial y}(x_0,y_0) = -14.7\ {}^\circ C/cm$$

$$T(x,y) \approx 36.7 - 14.7(x-2) - 29.4(y-1)$$

for (x,y) near $(1,2)$.

Now $\vec{r}(t) = \langle 2+6t, 1-t \rangle$

$$\frac{dx}{dt} = 6 \qquad \frac{dy}{dt} = -1$$

$$T(\vec{r}(t)) \approx 36.7 - 14.7 \cdot 6t - 29.4 \cdot (-t)$$

$$\frac{d}{dt} T(\vec{r}(t)) = -14.7 \cdot 6 - 29.4 \cdot (-1)$$

$$= -58.8\ {}^\circ C/s$$

What are these pieces?

$$\left.\frac{d}{dt}\right|_{t=0} T(\vec{r}(t)) =$$

$$\frac{\partial T}{\partial x}(2,1) \cdot \frac{dx}{dt}(0) + \frac{\partial T}{\partial y}(2,1) \cdot \frac{dy}{dt}(0)$$

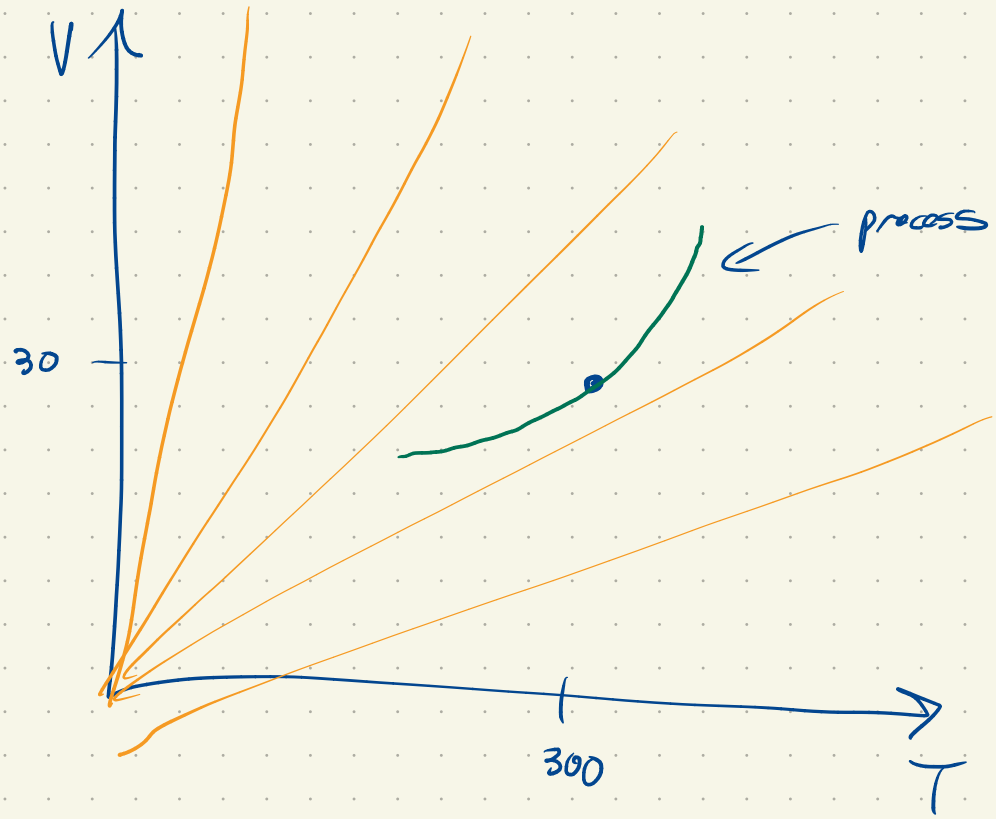
$$= (-14.7) \cdot 6 + (-29.4) \cdot (-1)$$

In general:

$$f(x,y) \qquad x = g(t) \quad y = h(t)$$

$$\frac{d}{dt} f(g(t), h(t)) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

e.g.



$P = 8.2$

$R = 0.082 \dfrac{L\ atm}{K\ mol}$

$$P = \frac{0.082\,T}{V}$$

What is the rate of change of pressure if T is changing at $10\,°K/h$ and V is changing at $4\ l/h$

$\dfrac{dT}{dt} = 10 \qquad \dfrac{dV}{dt} = 4$

$$\frac{\partial P}{\partial T}\frac{dT}{dt} + \frac{\partial P}{\partial V}\frac{dV}{dt} = \frac{0{,}082}{30}\cdot 10 - \frac{0{,}082}{30^2}\cdot 300\cdot 4$$

$$= 0{,}027 - 0{,}0109$$

$$= 0{,}0164 \text{ atm/hour}$$

Last class: chain rule

given

$$f(x,y), \quad x(t), \quad y(t)$$

$$\frac{d}{dt} f(x(t), y(t)) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$T(x,y) = x^2 e^{-y}$$

$$\vec{r}(t) = \langle t, t^2 \rangle$$

$$\frac{d}{dt} T(\vec{r}(t)) = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt}$$

e.g. compute

$$\frac{dT}{dt} \text{ at } t = \frac{1}{2}$$

$$\frac{\partial T}{\partial x} = 2x e^{-y} \qquad \frac{\partial T}{\partial y} = -x^2 e^{-y}$$

$$x = \frac{1}{2} \qquad y = \frac{1}{4}$$

$$\frac{\partial T}{\partial x} = 2 \cdot \frac{1}{2} \cdot e^{-1/4} \qquad \frac{\partial T}{\partial y} = -\frac{1}{4} e^{-1/4}$$

$$\frac{dx}{dt} = 1 \qquad \frac{dy}{dt} = 2t$$

$$\left.\frac{dy}{dt}\right|_{t=1/2} = 1$$

so $$\frac{dT}{dt} = e^{-1/4} \cdot 1 - \frac{1}{4} e^{-1/4} \cdot 1$$

$$= e^{-1/4}\left(\frac{3}{4}\right)$$

$$\approx 0.5841$$

But you could just do

$$T(x(t), y(t)) = t^2 e^{-t^2}$$

$$\frac{d}{dt} T(\vec{r}(t)) = 2te^{-t^2} + t^2(-2te^{-t^2})$$

at $t = \frac{1}{2}$

$$= e^{-1/4} - \frac{1}{4} e^{-1/4}$$

$$= e^{-1/4} \cdot \frac{3}{4} \quad \text{again.}$$

[value is mostly theoretical]

What if we add in another variable?

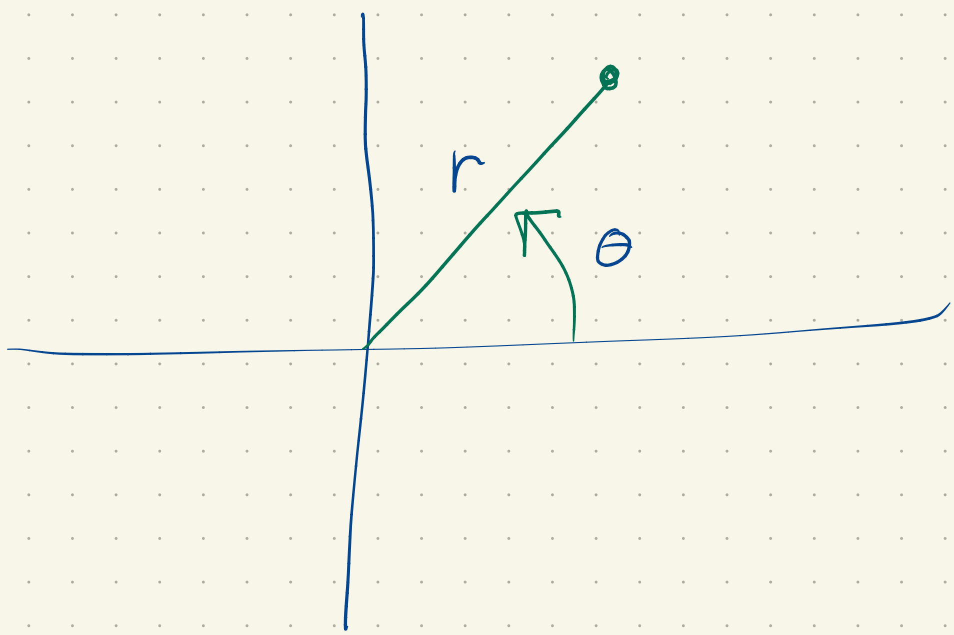
$$T(x, y, z) \qquad \vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$

$$\frac{d}{dt} T(\vec{r}(t)) = \frac{d}{dt} T(x(t), y(t), z(t))$$

$$= \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} + \frac{\partial T}{\partial z}\frac{dz}{dt}$$

What if you have a curve and I tell you not $x(t)$ and $y(t)$ but instead $\theta(t)$ and $r(t)$

$r(t)$ ↳ scalar, radius



$$x(r,\theta) = r\cos\theta$$

$$y(r,\theta) = r\sin\theta$$

(two output variables)

$$\frac{dx}{dt} = \frac{\partial x}{\partial r}\frac{dr}{dt} + \frac{\partial x}{\partial \theta}\frac{d\theta}{dt}$$

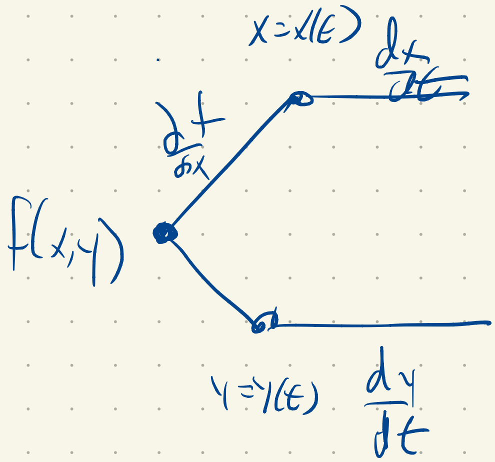
$$= \cos\theta\frac{dr}{dt} - r\sin\theta\frac{d\theta}{dt}$$

$$\boxed{\left(\begin{array}{ll} \theta = 0 & \frac{\partial r}{\partial t} \\ \theta = \frac{\pi}{2}, & -r\frac{d\theta}{dt} \end{array}\right)}$$

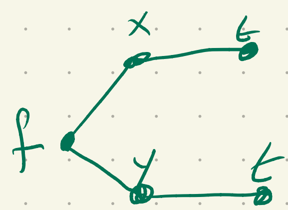
$$\frac{dy}{dt} = \sin\theta \frac{dr}{dt} + r\cos\theta \frac{d\theta}{dt}$$

Chain rule applies to each in turn.

So we'll focus on the case of one output variable.



$$\frac{d}{dt} f = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$



What if f depends on x, y but

$$x = x(u,v)$$

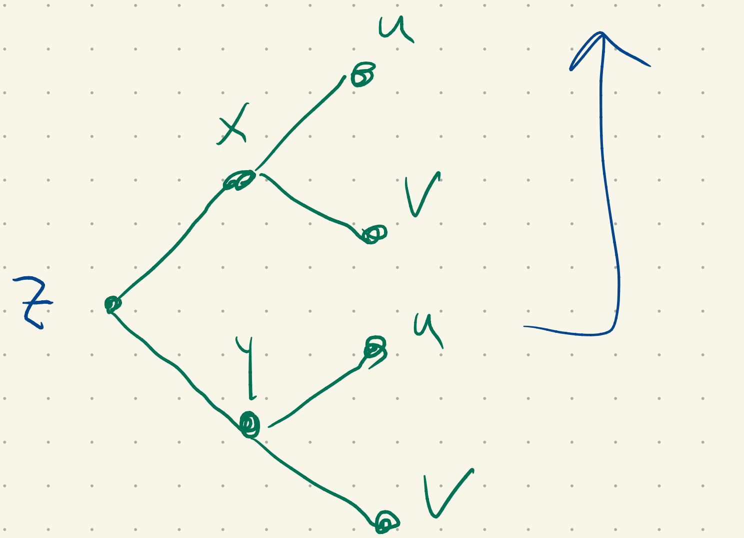
$$y = y(u,v)$$

?

$$z = f(x(u,v), y(u,v))$$

$$\frac{\partial z}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$

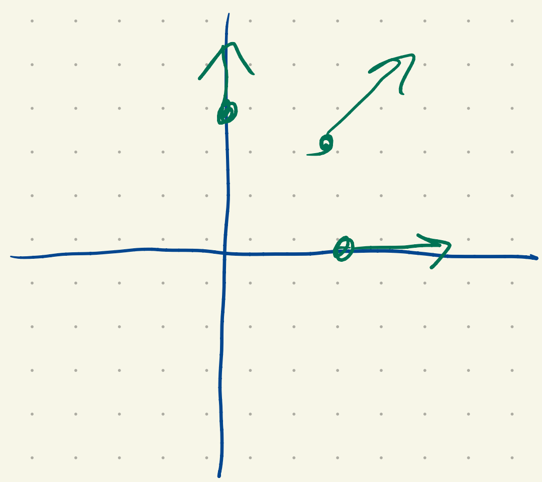
$$\frac{\partial z}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$$



$$h(x,y) = xy$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$



$$\frac{\partial h}{\partial r} = \frac{\partial h}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial h}{\partial y}\frac{\partial y}{\partial r}$$

$$= y\cos\theta + x\sin\theta$$

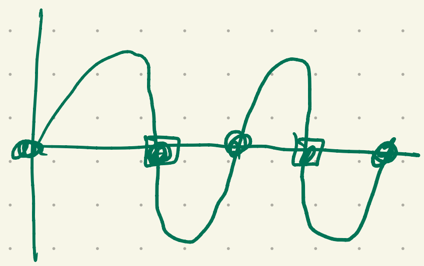
$$= r\sin\theta\cos\theta + r\cos\theta\sin\theta$$

$$= 2r\sin\theta\cos\theta$$

$$= r\sin(2\theta)$$

$$h(r,\theta) = r^2\cos\theta\sin\theta$$

$$= \frac{r^2}{2}\sin(2\theta)$$



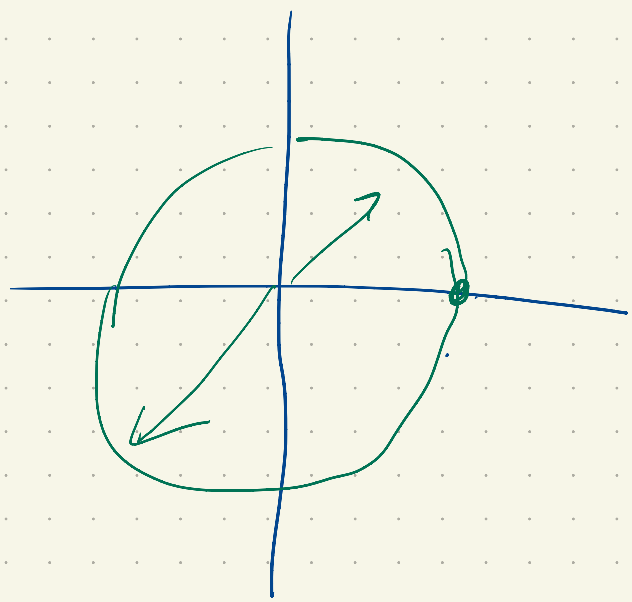
$$\frac{\partial h}{\partial \theta} = \frac{\partial h}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial h}{\partial y}\frac{\partial y}{\partial \theta}$$

$$= y(-r\sin\theta) + x\, r\cos\theta$$

$$= -r^2\sin^2\theta + r^2\cos^2\theta$$

$$= r^2(\cos^2\theta - \sin^2\theta)$$

$$= r^2\cos(2\theta)$$



r increases, goes like r^2

on $\theta = 0$, goes like r^2

on $\theta = \frac{\pi}{4}$ $\cos(\pi/2) = 0$,

$$\frac{\partial h}{\partial \theta} = 0$$

More variables:

$$w = f\left(x(s,t,u),\ y(s,t,u),\ z(s,t,u)\right)$$

$$\frac{\partial w}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u}$$

etc.