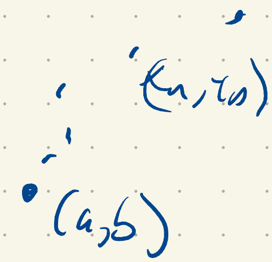
Last class

$$f(x,y) = \frac{xy}{x^2+y^2}$$



Claim $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ DNE

For us: $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$ if

whenever $\begin{matrix} x_n \to a \\ y_n \to b \end{matrix} \Rightarrow z_n = f(x_n, y_n) \to L$

(no matter what sequence)

$x_n = \frac{1}{n}$, $y_n = 0$ $\qquad z_n = f(x_n, y_n) = 0 \to 0.$

But $x_n = \frac{1}{n}$, $y_n = \frac{1}{n}$ $\qquad z_n = \dfrac{(1/n)^2}{2(1/n)^2} = \frac{1}{2} \to \frac{1}{2}$

So no limit.

But: $f(x,y) = \frac{xy}{\sqrt{x^2+y^2}}$ we claim

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0$$

$x = r\cos\theta$ $y = r\sin\theta$ $f(x,y) = \frac{r^2\cos\theta\sin\theta}{r} = \frac{r}{2}\sin(2\theta)$

So $|f(x,y)| \le \frac{r}{2} = \frac{1}{2}\sqrt{x^2+y^2}$

If $x_n \to 0$ and $y_n \to 0$ then $r_n = \sqrt{x_n^2+y_n^2} \to 0$.

$|z_n| = |f(x_n,y_n)| \le \frac{1}{2} r_n$ So

$-\frac{1}{2} r_n \le z_n \le \frac{1}{2} r_n$ and $z_n \to 0$ (Squeeze Thm)

Show plot.

$x = y^2$ $\qquad \dfrac{y^4}{y^4 + y^4} = \dfrac{1}{2} \quad (y \neq 0)$

$(x_n, y_n) = \left(\dfrac{1}{n^2}, \dfrac{1}{n} \right) \qquad f(x_n, y_n) = \dfrac{1}{2}$

Continuity:

We say $f(x,y)$ is cts at (a,b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b).$$

cts ⇒ continuous on all domain

It's a question of approximation.

$(x_n, y_n) \to (a,b)$

$f(x_n, y_n) \to f(a,b)$

error on inputs small ⇒ error in output small.

Continuous functions: (of x,y)

1) constants,

2) x

3) y

4) sums, products, differences of cts functions

$$f(x,y) = xy$$

$$f(x,y) = 1+xy$$

$$f(x,y) = 1+7xy$$

4') polynomials in x,y

5) old friends: sin, cos, ln, exp, arctan

on their domains

6) quotients $\dfrac{f(x,y)}{g(x,y)}$ on domain $\left(g(x,y) \neq 0! \right)$

7) rational functions $\dfrac{p(x,y)}{q(x,y)}$

Partial Derivatives

Mechanically.

$$f(x,y) = e^{x}\cos(x^3y^2)$$

To compute $\frac{\partial f}{\partial x}$ pretend y is a constant and

take a derivative with respect to x.

$$\frac{\partial f}{\partial x} = e^{x}\cos(x^3y^2) - e^{x}\sin(x^3y^2)\cdot 3x^2y^2$$

This is called the partial derivative of f with respect to x.

Ditto, $\frac{\partial f}{\partial y} = -e^{x}\sin(x^3y^2)\cdot(2yx^3)$

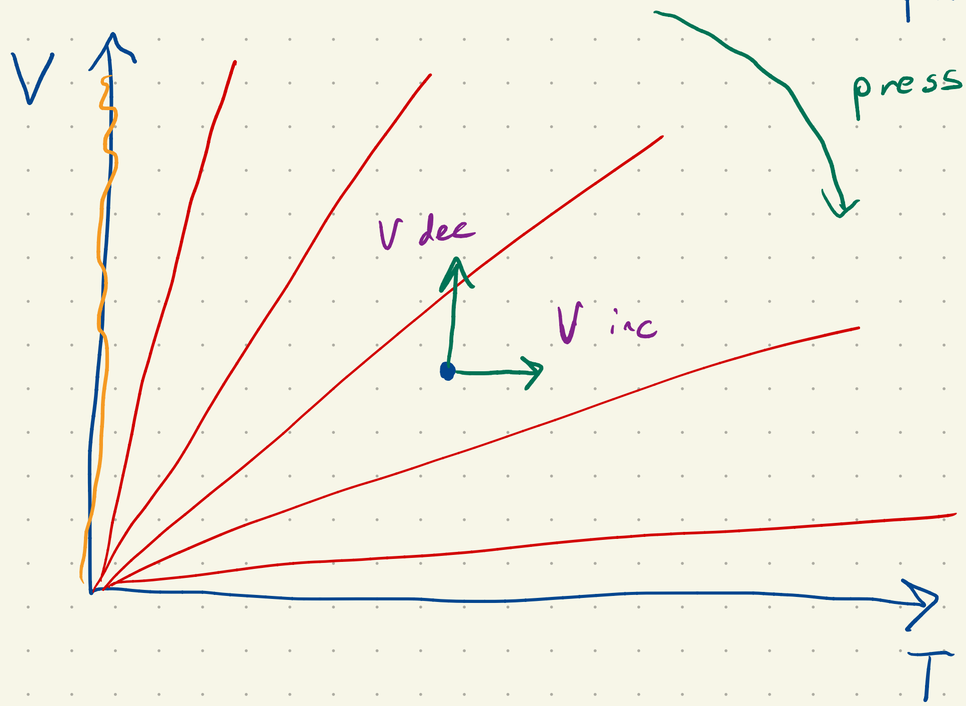
That's all there is to computing partial derivatives. I need to explain what on earth these mean.

$$V = \frac{nR}{P} T$$

$P = nRT/V$ as before

P const, V-T linear

$P = 0$, slope ∞. $P = \infty$, slope 0



level sets are lines

$$R = 0.082 \frac{L\,atm}{K\,mol}$$

1 mol gas, 30 l, 300K

$$P = 0.082 \frac{T}{V}$$

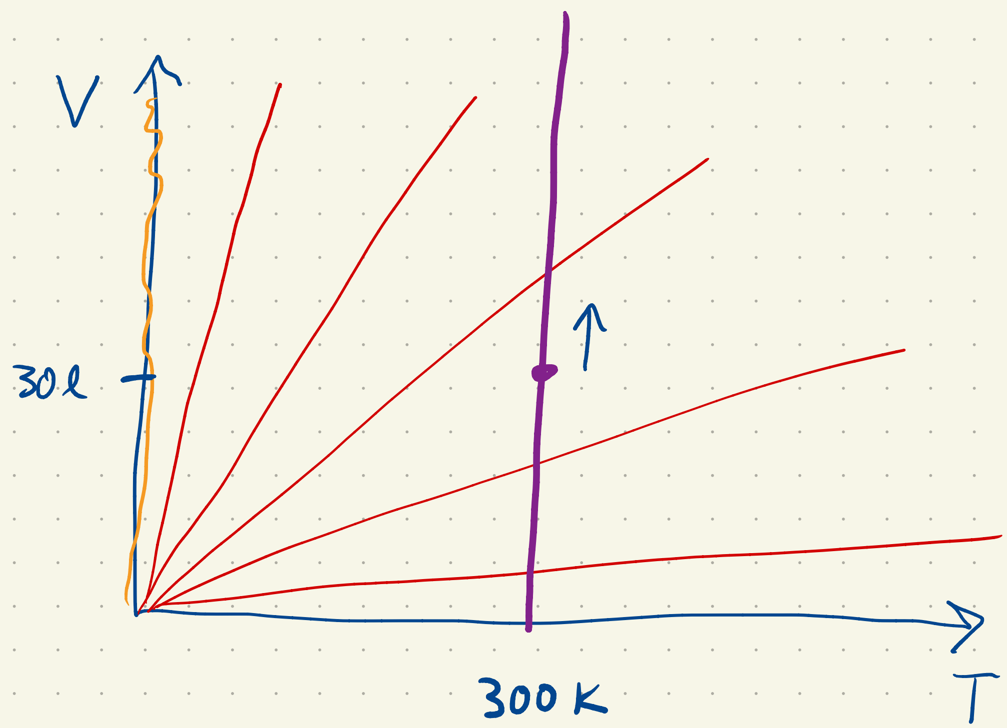
$[T] = K$
$[V] = l$

e.g. $T = 300\,K$, $V = 30\,l$

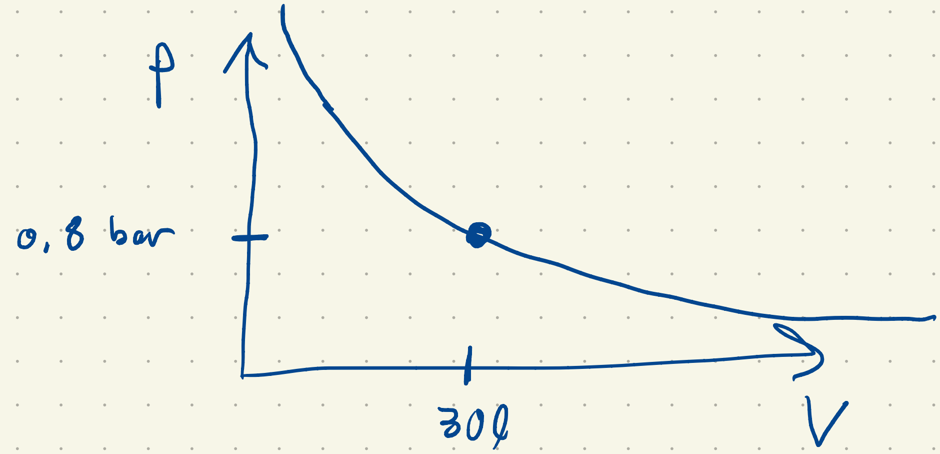
$P = 0.82$ atm

Now suppose you leave temperature fixed but monkey with the volume.

$$P = \frac{0.082 \cdot (300)}{V}$$



Pressure is now a function of V alone.



And we can ask "how does the pressure change as V changes?

$$\frac{dP}{dV} = -\frac{0.082 \cdot (300)}{V^2}$$

at $V = 30\,\ell$, $\frac{dP}{dV} = -0.027$ atm/ℓ

at $V = 20\,\ell$, $\frac{dP}{dV} = -0.0615$ atm/ℓ (pressure decreases faster)

Now put the T back in

$$\frac{\partial P}{\partial V} = -\frac{0.082\,T}{V^2}$$

Means " At temperature T and volume V,

if the volume increases the pressure

changes at a rate of

$-\frac{0.082\,T}{V^2}$ atm/ℓ."

It's called a partial derivative because it only talks about how a function changes when only <u>one</u> of the inputs is changed.

$$\left.\frac{\partial P}{\partial V}\right|_{\substack{T=300\\V=30}} = -0.027 \ \text{atm}/\ell$$

What if $T=300K$, $V=30\ell$ and we want to decrease the pressure by 0.1 atm. How much should we increase the volume?

Rough estimate: $-0.027 \cdot \Delta V = -0.1$

$$\Delta V = 3.7\ell$$

Check: $0.082 \cdot \frac{300}{33.7} = 0.73$ ish.

It was 0.82. This is close to what we want.

Given our one mol of gas, at what rate does the pressure change if the temperature is increased if $T = 300\,K$ and $V = 30\,\ell$?

$$\frac{\partial P}{\partial T} = \frac{0.082}{V}$$

$$\left.\frac{\partial P}{\partial T}\right|_{\substack{T=300 \\ V=30}} = 0.0027 \ \text{atm}/K$$

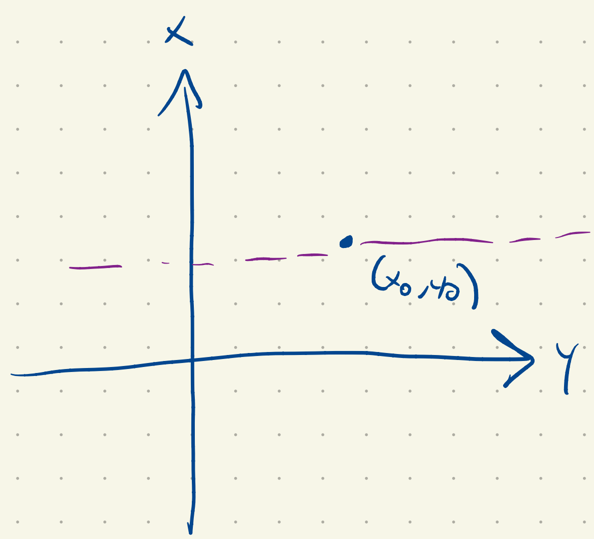
Summary

Given $f(x,y)$

$\frac{\partial f}{\partial x}(x_0, y_0)$ is the

rate of change of f w.r.t. x

if $x = x_0$, $y = y_0$



Just like ordinary derivatives, we have higher order partial derivatives

$f(x,y)$

second partial derivatives:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial}{\partial x} f = f_{xx}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\frac{\partial}{\partial y} f = f_{yx}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\frac{\partial}{\partial x} f = f_{xy}$$

Pure magic:

$$f(x,y) = x^3y^2 - x\ln(y)$$

$$f_x = 3x^2y^2 - \ln(y)$$

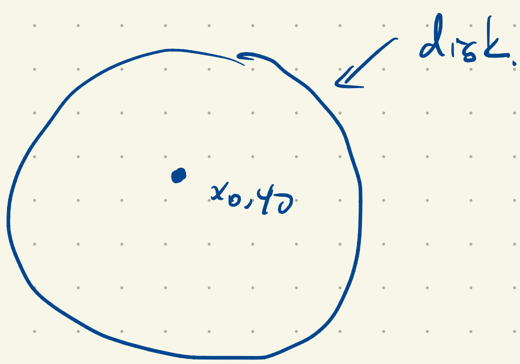
$$f_{xy} = 6x^2y - 1/y$$

$$f_y = 2x^3y - x/y$$

$$f_{yx} = 6x^2y - 1/y$$

$$f_{xy} = f_{yx} \quad (!)$$

The fact that this happens generally is called Clairault's Theorem and it has leegolese.



If f_{xy} and f_{yx} are defined here and cts then $f_{xy} = f_{yx}$ at (x_0, y_0)

Maxwell's Equations (2 of 'em)

E_1, E_2, E_3 components of electric field

B_1, B_2, B_3 magnetic field,

$$\frac{\partial E_1}{\partial x} + \frac{\partial E_2}{\partial y} + \frac{\partial E_3}{\partial z} = \frac{1}{\varepsilon_0}\rho$$ where

ρ is charge density. (Coulombs / m^3)

$$\frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y} + \frac{\partial B_3}{\partial z} = 0$$

By the end of the semester we'll know where these come from and what they mean.

These are examples of partial differential equations.

These govern numerous processes

- seismic vibrations
- ice flow in glaciers, ice sheets
- fluid flow (weather, ocean flow)
- E & M
- gravity (g.r.)
- heat transfer

Any time you have a continuum and varies rates of these are inter related.

e.g. wave equation in 2-d.

$$u_{tt} = c^2 \left(u_{xx} + u_{yy} \right)$$

u: height of membrane.

c: speed of vibrations

$$u = \sin(2x)\sin(4y)\cos\left(\sqrt{20}\, t\right)$$

$u_{tt} = -20u$

$u_{xx} = -4u$ $\qquad u_{tt} = u_{xx} + u_{yy}$

$u_{yy} = -16u$