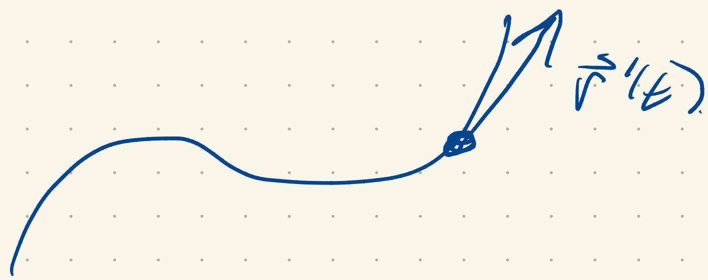
Section 11.4 Unit Normal + Tangent

Given $\vec{r}(t)$, $\vec{r}\,'(t)$ points in the direction of travel.



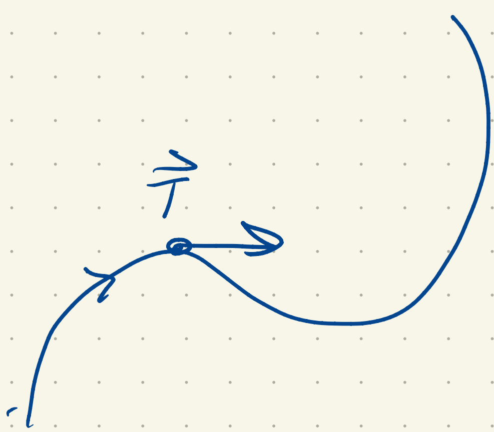
It encodes both direction + speed. $\|\vec{r}\,'(t)\|$ is speed.

It's sometimes useful to have a vector that points in the direction of travel but does not encode speed.

We'll use the unit vector:

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$$

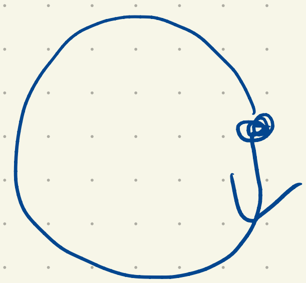
You can see $\vec{T}(t)$



e.g. $\vec{r}(t) = \langle \cos(\omega t), -\sin(\omega t) \rangle$

$$= \langle \cos(-\omega t), \sin(-\omega t) \rangle$$

counter clockwise!



$$\vec{r}'(t) = \omega \langle \sin(\omega t), -\cos(\omega t) \rangle$$

$$\|\vec{r}'(t)\| = \omega$$

$$\vec{T}(t) = \langle \sin(\omega t), -\cos(\omega t) \rangle$$

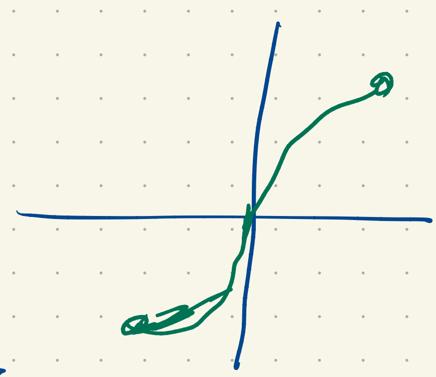
$$\vec{T}(0) = \langle 0, -1 \rangle$$

ω is the speed of traversal.

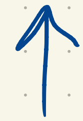
How about $\vec{r}(t) = \langle t^3, t \rangle$ $\qquad x = y^3$ (swap!)

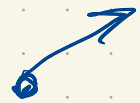
$$\vec{r}'(t) = \langle 3t^2, 1 \rangle$$

$$\|\vec{r}'(t)\| = \sqrt{9t^4 + 1}$$



$$\vec{T}(t) = \frac{1}{\sqrt{9t^4+1}} \langle 3t^2, 1 \rangle$$

$\vec{T}(0) = \frac{1}{1}\langle 0,1\rangle = \langle 0,1\rangle$ 

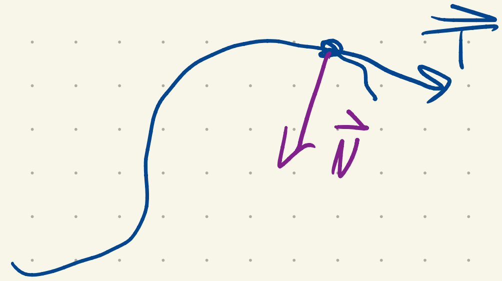
$T(1) = \frac{1}{\sqrt{10}}\langle 3,1\rangle$ 

$T(-1) =$  also!

$\vec{T}(t)\cdot\vec{T}(t) = 1$

$\frac{d}{dt}\vec{T}(t)\cdot\vec{T}(t) = 0$

So $\vec{T}'(t)$ is orthogonal to $\vec{T}$, We call it the <u>normal</u> vector.



But it might not be unit length.

$$\vec{N} = \frac{\vec{T}'}{\|\vec{T}'\|}$$

it tells you the direction $\vec{T}$ is turning to e.g. $\vec{r}(t) = \langle \cos(\omega t), -\sin(\omega t) \rangle$

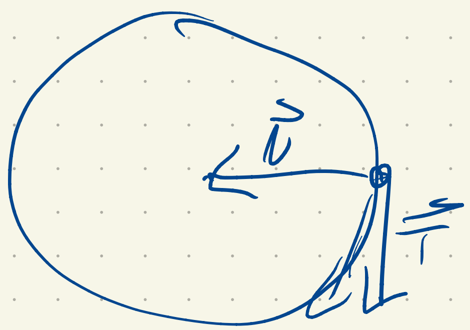
$$\vec{r}\,'(t) = \omega \langle -\sin(\omega t), -\cos(\omega t) \rangle$$

$$\vec{T}(t) = \langle -\sin(\omega t), -\cos(\omega t) \rangle$$

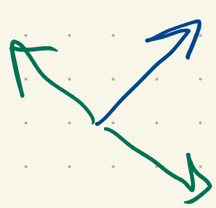
$$\vec{T}\,'(t) = \omega \langle -\cos(\omega t), \sin(\omega t) \rangle$$

$$\vec{N}(t) = \langle -\cos(\omega t), \sin(\omega t) \rangle$$

$$= -\vec{r}(t)$$



In the plane $\vec{T} = \langle t_x, t_y \rangle$



Only two possibilities for N.

$s(t) = \|\vec{r}'(t)\| = \text{speed}$

$$\vec{r}'(t) = s(t)\, T(t)$$

$$\vec{r}''(t) = s'(t)\, T + s\, T'(t)$$

$$= s'(t)\, T + s\, \|T'\| \frac{T'}{\|T'\|}$$

$$= s'(t)\, \vec{T} + s\, \|T'\|\, \vec{N}$$

Acceleration has two components one tangental and the other Normal.

Tangental component: $s'(t)$ how is the speed changes.

Normal is about turning instead.

$$\vec{r}''(t) \cdot \vec{T} = \underbrace{s'(t)}_{a_T}$$

$\vec{r}''(t) \cdot \vec{N} = a_N$ normal component of acceleration.

(usually, $\| \vec{r}'' - \vec{r}'' \cdot \vec{T} \|$) cuz $\vec{N}$ is a pain!

$\vec{r}(t) = \langle \cos(t^2), \sin(t^2) \rangle$

$\vec{r}'(t) = \langle -2t \sin(t^2), 2t \cos(t^2) \rangle$

$\vec{T}(t) = \langle -\sin(t^2), \cos(t^2) \rangle$

$\vec{r}''(t) = \langle -2 \sin(t^2), 2\cos(t^2) \rangle + \langle -4t^2 \cos(t^2), -4t^2 \cancel{\cos}(t^2) \rangle$

$\vec{T} \cdot \vec{r}'' = +2$

tangential component $a_T = 2$

$\vec{r}''(t) - \vec{T} \cdot \vec{r}'' \, \vec{T} = -4t^2 \langle \cos(t^2), \sin(t^2) \rangle$

$\| \vec{r}''(t) - \vec{T} \cdot \vec{r}'' \, \vec{T} \| = \underbrace{4t^2}_{a_N}$

$\vec{T}' = \langle -2t \cancel{\sin}(t^2), 2t \sin(t^2) \rangle$

$\vec{N} = \langle \cos(t^2), \sin(t^2) \rangle$