Section 13.4 (Acceleration, Velocity, Momentum, Force)

If $\vec{r}(t)$ describes position as a function of time

1) $\vec{v}(t) = \vec{r}\,'(t) = \frac{d}{dt}\vec{r}(t)$ is velocity

2) $|\vec{v}(t)| = |\vec{r}\,'(t)|$ is speed

3) $\vec{a}(t) = \vec{v}\,'(t) = \frac{d}{dt}\vec{v}(t) = \vec{r}\,''(t)$ is acceleration

e.g. If $\vec{r}(t) = \langle \sin(2t), \tan(t), 1-t \rangle \quad -\frac{\pi}{2} < t < \frac{\pi}{2}$

$$\vec{v}(t) = \langle 2\cos(2t), \sec^2(t), -1 \rangle$$

$$\vec{a}(t) = \langle -4\sin(2t), 2\sec(t)\sec(t)\tan(t), 0 \rangle$$

$$|\vec{r}\,'(t)| = \sqrt{4\cos^2(2t) + \sec^4(t) + 1}$$

e.g. Suppose a particle has

acceleration

$$\vec{a}(t) = \langle -\cos(t), -\sin(t), -1 \rangle$$

and $\vec{r}(0) = \langle 5, 2, 2 \rangle$

$$\vec{r}\,'(0) = \langle 0, 1, 3 \rangle.$$

Determine $\vec{r}(t)$.

$$\vec{v}\,'(t) = \vec{a}(t)$$

$$\vec{v}(t) = \int \vec{a}(t)\,dt + \vec{C}$$

$$= \langle -\sin(t), \cos(t), t \rangle + \vec{C}$$

$$\vec{v}(0) = \langle 0, 1, 0 \rangle + \vec{C}$$

$$\langle 0, 1, 3 \rangle = \langle 0, 1, 0 \rangle + \overbrace{\langle 0, 0, 3 \rangle}^{\vec{C}}$$

$$\vec{v}(t) = \langle -\sin(t), \cos(t), 3 + t \rangle$$

$$\vec{r}\,'(t) = v(t)$$

$$\vec{r}(t) = \int \vec{v}(t)\,dt + \vec{c}_2$$

$$= \langle \cos(t), \sin(t), 3t + \tfrac{t^2}{2} \rangle + \vec{c}_2$$

$$\langle 5, 2, 2 \rangle = \langle 1, 0, 0 \rangle + \vec{c}_2$$

$$\vec{c}_2 = \langle 4, 2, 2 \rangle$$

$$\vec{r}(t) = \langle 4 + \cos(t), 2 + \sin(t), 2 + 3t + \tfrac{t^2}{2} \rangle$$

We reconstruct position from acceleation + two data points (initial position, velocity)

Newton 2:

$\vec{p}$: momentum (total quantity of motion)

$\vec{F}$: force

If object has mass m and velocity $\vec{v}$

$$\vec{p} = m\vec{v} = m\vec{r}\,'$$

The rate of change of momentum is force.

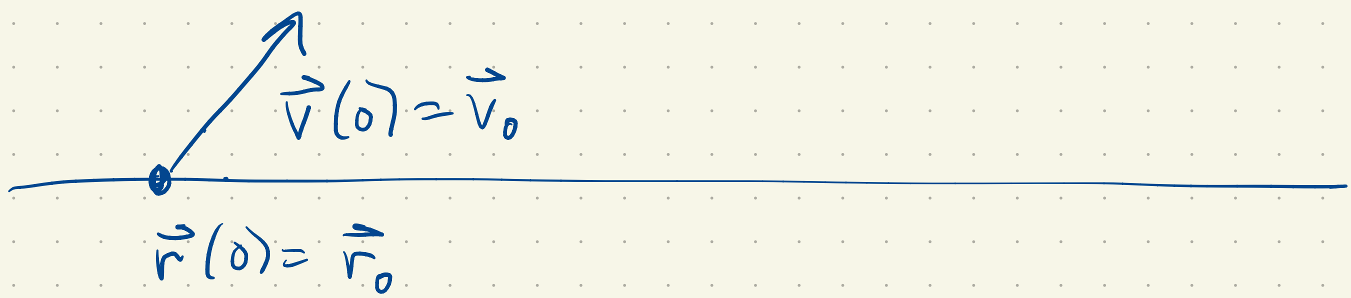
$$\frac{d}{dt}\vec{p} = \vec{F}$$

m constant: $\quad m\vec{r}\,'' = \vec{F} \qquad (\vec{F} = m\vec{a})$

If you know the force acting on an object, you know the acceleration:

$$\vec{a} = \frac{1}{m}\vec{F}.$$

And if you know initial position and velocity then you can reconstruct the position.



Projectiles close to earth:

$$\vec{F}_g = -9.8\,\hat{k} \quad m/s^2$$

$$\vec{v}(t) = \int -9.8\,\hat{k}\,dt + \vec{C}_1$$

$$= -9.8t\,\hat{k} + \vec{C}_1$$

$$\vec{v}(0) = \vec{C}_1$$

$$\vec{v}(t) = -9.8t\,\hat{k} + \vec{v}_0$$

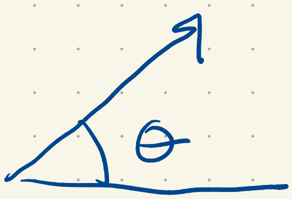
$$\vec{r}(t) = \int \vec{v}(t)\,dt + \vec{C}_2$$

$$\vec{r}(t) = \frac{-9.8t^2}{2}\hat{k} + \vec{v}_0 t + \vec{C}_2$$

$$\vec{r}(0) = 0 + 0 + \vec{C}_2$$

$$\vec{r}(t) = \vec{r}_0 + t v_0 - \frac{9.8}{2}t^2 \hat{k}$$

$$\left(9.8 \to 0 \Rightarrow \text{linear motion!} \right)$$



$$\vec{r}_0 = \vec{0}$$

$$\vec{v}_0 = v_0 \cos(\theta)\,\hat{i} + v_0 \sin\theta\,\hat{k}$$

$$\vec{r}(t) = v_0 \cos\theta\, t\,\hat{i} + \left[v_0 \sin\theta t - \frac{9.8}{2}t^2\right]\hat{k}$$

This is a parabolic trajectory.

When is $z = 0$?
$$t\left[v_0 \sin\theta - \frac{9.8}{2} t\right] = 0$$

$t = 0$ or

$$t = \frac{2v_0}{9.8} \sin\theta$$

e.g. $v_0 = 150\, m/s$ $\quad \theta = \pi/4 = 45°$

How far when strikes ground?

$$t = \frac{300}{9.8} \frac{1}{\sqrt{2}} \simeq 21.64$$

$$x = v_0 \cos\theta\, t$$

$$= 150 \frac{1}{\sqrt{2}} \frac{300}{9.8} \frac{1}{\sqrt{2}} = \frac{150 \cdot 150}{9.8} \simeq 2295\, m$$

Peak height?

$$\left[v_0 \sin\theta t - \frac{9.8}{2} t^2\right] = z(t)$$

$$z'(t) = v_0 \sin\theta - 9.8t$$

$$z'(t) = 0 \Rightarrow \quad t = \frac{v_0 \sin\theta}{9.8}$$

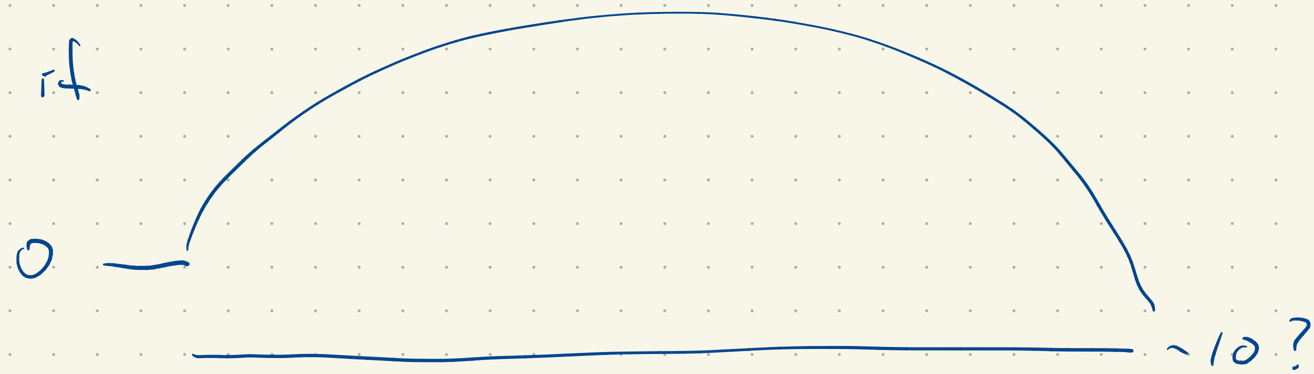
So peak happens at $t = \dfrac{150 \cdot \sin(45^\circ)}{9.8} = 10.82\,s$

$z(10.82) = 574$ m

$$\frac{(v_0 \sin\theta)^2}{9.8} - \frac{9.8}{2} \frac{(v_0 \sin\theta)^2}{(9.8)^2}$$

$$= \frac{1}{2} \frac{(v_0 \sin\theta)^2}{9.8} = 573$$

What if



$$t\left[v_0 \sin\theta - \frac{9.8}{2} t\right] = -10$$

$$-\frac{9.8}{2} t^2 + t \frac{150}{\sqrt{2}} + 10 = 0$$

$$t = 21.74$$

$$x \approx 2306$$