$$\int |\varphi - \tilde{s}|^p \le \left(\frac{\varepsilon^p}{2^p M^p}\right) 2^p M^p = \varepsilon^p$$

$$\|\varphi - \tilde{s}\|_{L^p(I)} < \varepsilon.$$

$$\|f - \tilde{s}\| < 3\varepsilon.$$

Exercise: Find the continuous g that works.

$$1 \in L^\infty$$

integrable simple functions are zero on a set of infinite measure.

$$\|\varphi - 1\|_\infty \ge 1$$

simple functions are dense in L^∞.

Basic construction. $f \in L^\infty$ WLOG $|f| \le \|f\|_\infty$ everywhere

$$\varphi_n \to f \text{ uniformly.} \Rightarrow \varphi_n \to f \text{ in } L_\infty.$$

$$f_n \to f \text{ in } L_\infty$$

$$\|f_n - f\|_\infty \to 0$$

$\forall \varepsilon > 0$ there exists N so $n \ge N \Rightarrow \|f_n - f\|_\infty < \varepsilon$

$$\inf\{M : |f_n - f| \le M \text{ a.e.}\} < \varepsilon \quad \text{if } n \ge N$$

$$f_n \to f \text{ uniformly} \Rightarrow f_n \xrightarrow{L_\infty} f$$

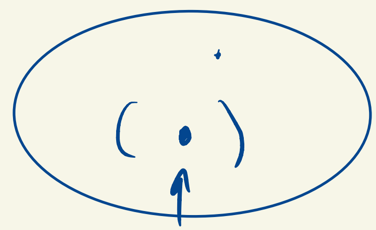
Suppose $\{f_n\}$ are continuous and bounded on $\mathbb{R}$

$$f_n \xrightarrow[L^\infty]{} f$$

I claim f is continuous.

$$\|f_n - f_m\|_\infty = \sup_{x \in \mathbb{R}} |f_n(x) - f_m(x)|$$

$\leq$ is free.



$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty \quad \text{a.e.}$$

but by continuity, $|f_n(x) - f_m(x)| \leq \|f_n - f_m\|$ everywhere.

$f_n \xrightarrow{L_\infty} f \implies$ Cauchy in L_∞

$\implies$ Cauchy in $\underbrace{C(\mathbb{R}) \cap B(\mathbb{R})}_{\text{complete}}$ $\quad C_b(\mathbb{R})$

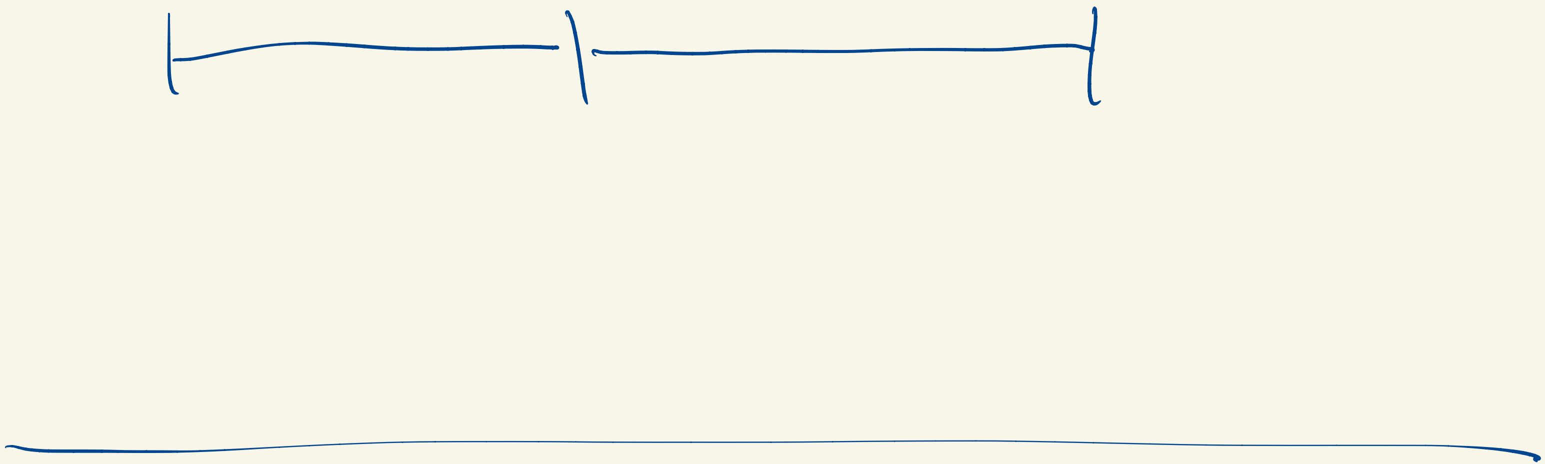
$\implies f_n \xrightarrow{C_b} g$ for some g

$\implies f_n \xrightarrow{L_\infty} g$

By uniqueness of limits $f = g$.

$\implies f$ is continuous.

$$\overline{C_b(\mathbb{R})}^{L^\infty} = C_b(\mathbb{R})$$



Def: We say that measurable functions $f_n \to f$ *in measure* if for all $\varepsilon > 0$ there exists N so that if $n \geq N$ $m(\{|f_n - f| \geq \varepsilon\}) < \varepsilon$.

Very weak notion of convergence.

Claim: If $f_n \to f$ in L_1 then $f_n \to f$ in measure.

Let $\varepsilon > 0$.

$$\int |f_n - f| \geq \varepsilon \, m(\{|f_n - f| \geq \varepsilon\})$$

$$\frac{1}{\varepsilon} \underbrace{\|f_n - f\|_1}_{} \geq m(\{|f_n - f| \geq \varepsilon\}).$$

↳ take n so large that $\|f_n - f\|_1 < \varepsilon^2$

Exercise: The same holds for $1 \leq p < \infty$.

L_p convergence $\Rightarrow$ convergence in measure $\not\Rightarrow$ L_p conv. 1

p.w. a.e. $\not\Rightarrow$ conv. in measure $\not\Rightarrow$ p.w. conv a.e.

2 3

But if $m(D) < \infty$ $(f_n : D \to \overline{\mathbb{R}})$

p.w. a.e. $\Rightarrow$ conv. in measure

(limit finite a.e.)

1 rising typewriter bumps. (scale each bump so $\int |f_n| = 1$)

Typewriter bumps converge in measure to 0.

$\varepsilon > 0$ $\qquad 2^{-n} < \varepsilon$

$$m(\{ |f_m - 0| > \varepsilon \}) = 2^{-n} < \varepsilon$$

2

$f_n = \chi_{[n,\infty)}$ $\qquad f_n \to 0$ p.w. (everywhere)

$$f_n \not\to 0 \text{ in measure.}$$

$$m(\{ |f_n - 0| > \varepsilon \}) = \infty$$

3 typewriter bumps $\to 0$ in measure.

but not pointwise a.e.

Recall Egoroff's Thm:

f_n meas, $f_n : D \to \overline{\mathbb{R}}$, $f_n \to f$ p.w. a.e.

↳ finite a.e.

$m(D) < \infty$

$\forall \varepsilon > 0 \ \exists \ E \subseteq D, \ m(E) < \varepsilon$, $f_n \to f$ uniformly on $D \setminus E$.

almost uniform convergence $\Rightarrow$ convergence in measure.

↳ domain of finite measure.

If $f_n \to f$ on a bounded domain and f is finite a.e.

then $f_n \to f$ almost uniformly $\Rightarrow$ $f_n \to f$ in measure.

A sequence is Cauchy in measure if for all $\varepsilon > 0$ there exists N so $n, m \geq N$ then

$$m(\{|f_n - f_m| \geq \varepsilon\}) < \varepsilon.$$

Thm: If (f_n) is Cauchy in measure then there is a limit f such that $f_n \to f$ in measure and a subsequence $f_{n_k} \to f$ pointwise a.e.

Exercise: Convergence in measure implies Cauchy in measure.

Cor: If $f_n \to f$ in L_p $1 \le p < \infty$

then there is a subsequence $f_{n_k} \to f$ p.w. a.e.

Pf: $f_n \xrightarrow{L_p} f \Rightarrow f_n \xrightarrow{\text{in meas.}} f \Rightarrow f_{n_k} \xrightarrow{\text{p.w. a.e.}} f$

Cor: L^p is complete.

Suppose $\{f_n\}$ is Cauchy in L^p.

Exercise: $\{f_n\}$ is Cauchy in measure.

$\Rightarrow$ $f_{n_k} \to f$ p.w. a.e. for some f.

$f_n \to f$ in measure.

candidate limit.

Claim: $f \in L^p$

$$|f_{n_k}|^p \to |f|^p \quad \text{p.w. a.e.}$$

Fatou:
$$\int |f|^p \leq \liminf \int |f_{n_k}|^p \leq \underbrace{\sup_n \|f_n\|_p^p < \infty}_{\text{Cauchy implies bounded.}}$$

So $f \in L^p$.

$$\int |f - f_{n_k}|^p \leq \liminf_{j \to \infty} \int |f_{n_j} - f_{n_k}|^p$$

$$= \liminf_{j \to \infty} \underbrace{\|f_{n_j} - f_{n_k}\|_p^p}_{< \varepsilon \text{ for } k, j \text{ big enough.}}$$

(Cauchy!)

$f_{n_k} \to f$ in L^p.

Cauchy + conv. subsequence $\Rightarrow$ convergence.