L^p Hölder's Ineq.

$L^p = \{ f, \text{meas}, \int_E |f|^p < \infty \}$

$1 \leq p < \infty \qquad \|f\|_p = \left(\int_E |f|^p \right)^{1/p}$

$L^p(E)$

$1 < p < \infty \qquad \frac{1}{p} + \frac{1}{q} = 1$

$f \in L^p(E),\ g \in L^q(E) \Rightarrow fg \in L^1(E)$

$$\int_E |fg| \leq \|f\|_p \|g\|_q$$

Suppose $m(E) < \infty$ $\qquad L^p(E)$ $\qquad 1 < p < \infty$

$$\int_E |f| = \int_E |f \cdot 1| \le \|f\|_{L^p(E)} \underbrace{\|1\|_{L^q(E)}}_{\substack{< \infty \\ (m(E))^{1/q}}}$$

f_α decay

g_β sing.

$$L^p(E) \subseteq L^1(E)$$

$$L^p(E) \subseteq L^p(\mathbb{R})$$

$\uparrow$ $\qquad \Rightarrow$

If $1 \le p_1 < p_2 < \infty$

$$\underbrace{L^{p_2}(E) \subseteq L^{p_1}(E)}$$

$(m(E) < \infty)$

$p_1 < p_2$

$1 < p_2/p_1$

$f \in L^{p_2}$

$$\int_E |f|^{p_1} \le \| \, |f|^{p_1} \, \|_{L^{p_2/p_1}(E)} \|1\|_{L^r(E)}$$

$$\frac{p_1}{p_2} + \frac{1}{r} = 1$$

$$= \left[\int \left(|f|^{p_1} \right)^{p_2/p_1} \right]^{p_1/p_2} \left(m(E) \right)^{1/r}$$

$$= \|f\|_{L_{p_2}(E)}^{p_1} \, m(E)^{1 - p_1/p_2}$$

If $m(E) = \infty$ then there are no inclusions between L^p spaces

(last class saw this for $E = \mathbb{R}$) f_α, g_β

$$\frac{1}{p} + \frac{1}{q} = 1 \qquad p = q(p-1)$$

(dual exponents)

$$\frac{p}{q} = p - 1$$

Lemma: If $f \in L^p$ then $|f|^{p-1} \in L^q$ and

$$\| \, |f|^{p-1} \|_{L^q} = \|f\|_p^{p-1} \qquad 1 < p < \infty.$$

Pf:

$$\int |f|^{(p-1)q} = \int |f|^p$$

$$\| |f|^{p-1} \|_{L^q}^q = \|f\|_{L^p}^p$$

$$\| \, |f|^{p-1} \|_{L^q} = \|f\|_{L^p}^{p/q} = \|f\|_{L^p}^{p-1}.$$

Thm (Minkowski's Inequality)

Suppose $1 < p < \infty$. If $f, g \in L^p$ then

$$\|f+g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}$$

(Similarly for $L^p(E)$)

Pf: Observe $(p>1)$

$$\|f+g\|_{L^p}^p = \int |f+g|^p = \int |f+g|^{p-1} |f+g|$$

$$\leq \int |f+g|^{p-1} (|f| + |g|)$$

$$= \int |f+g|^{p-1} |f| + \int |f+g|^{p-1} |g| .$$

From the previous Lemma, $|f+g|^{p-1} \in L_q$ with $\frac{1}{p}+\frac{1}{q}=1$.

So $$\int |f+g|^{p-1}|f| \leq \| |f+g|^{p-1} \|_{L^q} \|f\|_{L^p} \qquad \text{(Hölder!)}$$
$$= \| |f+g| \|_{L^p}^{p-1} \|f\|_{L^p} \qquad \text{(Lemma)}.$$

The same holds for g and we conclude

$$\| f+g \|_{L^p}^p = \int |f+g|^p \leq \| f+g \|_{L^p}^{p-1} \left(\|f\|_{L^p} + \|g\|_{L^p} \right).$$

So long as $\|f+g\|_{L^p} \neq 0$ we conclude

$$\|f+g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}.$$

The same inequality is trivial if $\|f+g\|_{L^p} = 0$.

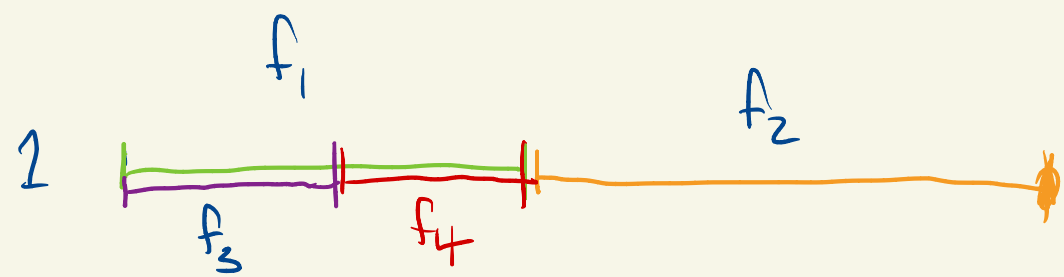
[Are the L^p spaces complete? (Friday)

Density theorems?

L^∞

$f_n \to f$ in L^1 $\overset{?}{\Rightarrow}$ $f_n \to f$ p.w. a.e.] No ☹

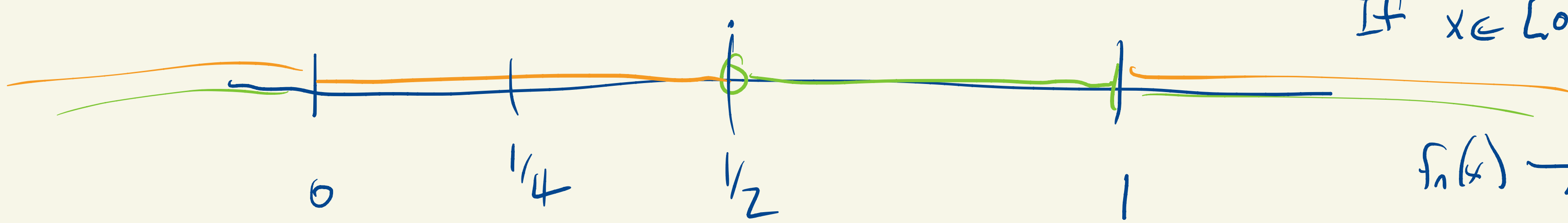
We can still recover something!



typewriter bumps.

$f_n \to 0$ in L^1

If $x \in [0,1]$



$f_n(x) \not\to 0$.

L^∞. $f \in L^\infty$ if f is meas. and $\exists M$ so $|f| \le M$ a.e.

$$\|f\|_\infty = \underbrace{\inf\{M : |f| \le M \text{ a.e.}\}}_{\text{ess. sup. of } f}$$

Is this a norm?

$\|f\|_\infty \ge 0$. $\|f\|_\infty = 0 \Rightarrow |f| \le 0$ a.e. $f = 0$ a.e.

Claim: If $f \in L^\infty$ then $|f| \le \|f\|_\infty$ a.e.

$E_n = \{|f| > \|f\|_\infty + \frac{1}{n}\}$ $m(E_n) = 0$

$E = \bigcup_n E_n$ $E = \{|f| > \|f\|_\infty\}$ null

E is null, $|f| \le \|f\|_\infty$ except on E

$|f| \le \|f\|_\infty$ a.e.

$\|cf\|_\infty = |c| \, \|f\|_\infty$ is easy

If $f, g \in L_\infty$ is $f+g \in L_\infty$?

$|f+g| \le |f| + |g| \le \|f\|_\infty + \|g\|_\infty$ a.e.

$\|f+g\|_\infty \le \|f\|_\infty + \|g\|_\infty$.

$\frac{1}{p} + \frac{1}{q} = 1$ $\quad p = 1, \; q = \infty$

$f \in L_1 \quad g \in L_\infty$

$\int |fg| \le \int |f| \, \|g\|_\infty = \|g\|_\infty \int |f| = \|g\|_\infty \|f\|_1$

Completeness: stand by.....

Approximation in L^p. $B(X) \wedge C(X)$

Claim: Integrable simple functions are dense in L^p $1 \leq p < \infty$
(and we can take the support of such function ~~to~~ be bounded)

Moreover, continuous functions with bounded support are also dense.

Pf. (Sketch)

Step 1: remove a tail. $\int_{\mathbb{R}\setminus[-n,n]} |f|^p \to 0$ as $n \to \infty$. (DCT)

$\tilde{f} = \chi_I f$ $I = [-n,n]$ where $\int_{\mathbb{R}\setminus I} |f|^p < \varepsilon$.

Step 2: approximate $\tilde{f}$ by integrable simple functions.

φ_n, $\quad |\varphi_n| \le |\tilde{f}|$, $\quad$ simple. (integrable)

support $\varphi_n \subseteq I$

$\underline{\varphi_n \to \tilde{f} \text{ pointwise.}}$ $\qquad |\tilde{f} - \varphi_n| \to 0$ p.w.

$$
\begin{aligned}
|\tilde{f} - \varphi_n|^p &\le 2^p \left(|\tilde{f}|^p + |\varphi_n|^p \right) \\
&\le 2^p \left(|\tilde{f}|^p + |\tilde{f}|^p \right) \\
&= \underbrace{2^{p+1} |\tilde{f}|^p}_{L^1}
\end{aligned}
$$

$|\tilde{f} - \varphi_n|^p \to 0$, dominated

$\int |\tilde{f} - \varphi_n|^p \to 0$ $\qquad$ $\|\tilde{f} - \varphi_n\|_{L^p} \to 0$

$\varphi = \varphi_n$ with n large enough so that

$$\|\tilde{f} - \varphi\|_{L^p} < \varepsilon.$$

$$\|f - \varphi\|_{L^p} < 2\varepsilon.$$

Now find a continuous function $\tilde{g}$ on I with

$$m(\{\tilde{g} \neq \varphi\}) < \frac{\varepsilon^p}{2^p M^p} \quad \text{where } |\varphi| \leq M \text{ everywhere.}$$

WLOG $|\tilde{g}| \leq M$.

$$\int |\varphi - \tilde{s}|^p \leq \left(\frac{\varepsilon^p}{2^p M^p}\right) 2^p M^p = \varepsilon^p$$

$$\|\varphi - \tilde{s}\|_{L^p(I)} < \varepsilon.$$

$$\|f - \tilde{s}\| < 3\varepsilon.$$

Exercise: Find the continuous g that works.

$$1 \in L^\infty$$

integrable simple functions are zero on a set of infinite measure.

$$\|\varphi - 1\|_\infty \geq 1$$