So $$\int |\varphi_n - f| = \int_{I^c} |\varphi_n - f| + \int_I |\varphi_n - f|$$

$$= \int_{I^c} |f| + \int |\varphi_n - \chi_I f|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$\square$

Pf of b)

Given $f \in L^1$ find an integrable simple function φ with bounded support (on some interval I, say) such that

$\|f - \varphi\|_1 < \varepsilon$.

Pick M such that $|\varphi| \le M$ everywhere.

From Borel's Theorem we can find a continuous function h on I with $m(\{h \neq \varphi\}) < \frac{\varepsilon}{2M}$ and with $|h| \le M$ everywhere. We extend h by 0 outside of I. Then

$$\boxed{h = \max(\min(\tilde{h}, M), -M)}$$

$$\|f - h\|_1 \le \|f - \varphi\|_1 + \|\varphi - h\|_1$$

$$< \varepsilon + \frac{\varepsilon}{2M} 2M$$

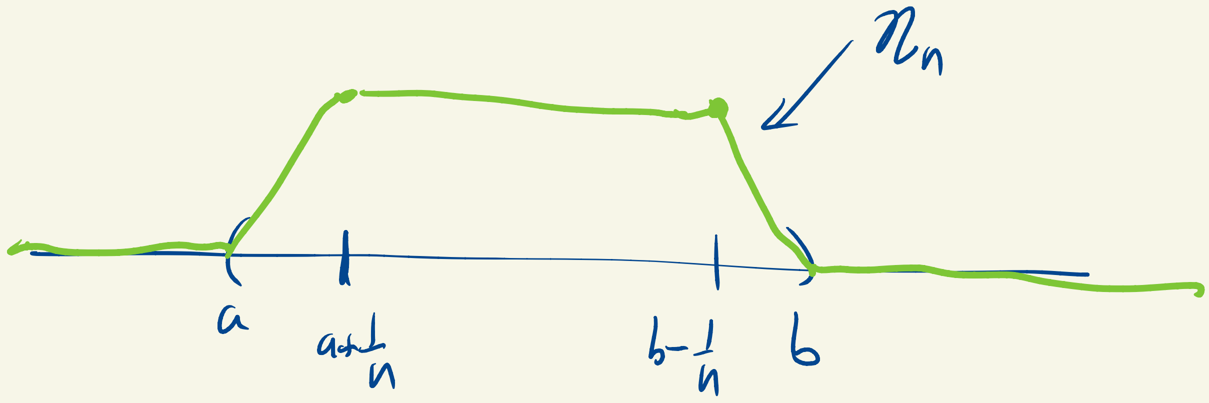
$$= 2\varepsilon.$$

$$\boxed{\begin{array}{l} |\varphi - h| \le 2M \\ m(\{f - \varphi \neq 0\}) < \frac{\varepsilon}{2M} \end{array}}$$

Now h need not be continuous.

Let $I = (a,b)$. For n sufficiently large define η_n

as follows



Observe that for each n, $\eta_n h = g_n$ is continuous.

Moreover $g_n \to h$ p.w. a.e. and $|g_n| \leq |h| \in L^1$.

$\int g_n \to \int h$

$|g_n - h|$ $\int |g_n - h| \to \int 0 = 0$

$\uparrow L^1$

So $g_n \to h$ in L^1.

In particular we can pick n large enough so that $\| h - g_n \| < \varepsilon$,

in which case $\| f - g_n \| < 3\varepsilon$.

L_1

stay tuned!

$L_p \qquad 1 \leq p < \infty$

$L_p = \{ f, \text{ measurable}, |f|^p \in L^1 \}$

(equivalence classes thereof)

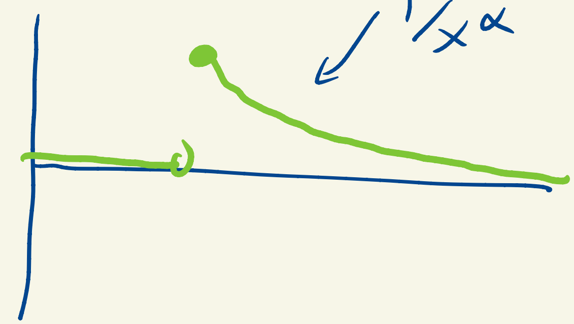
$$\|f\|_p = \left(\int |f|^p \right)^{1/p}$$

$L_p(E)$ with E measurable is defined similarly.

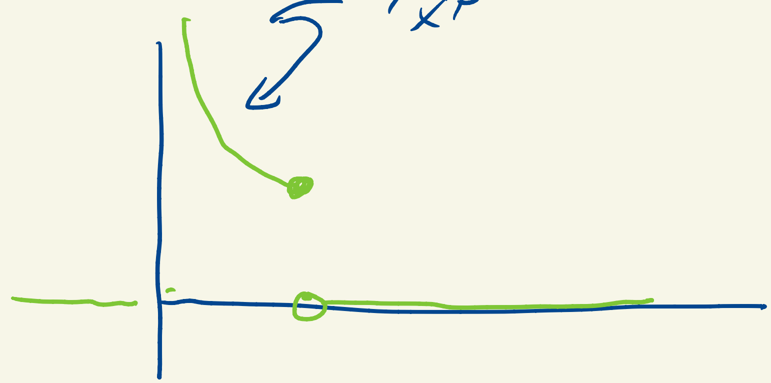
Two phenomena: decay at ∞

blowups at points

$$f_\alpha = \begin{cases} 1/x^\alpha & x \geq 1 \\ 0 & \text{otherwise} \end{cases}$$



$$g_\beta = \begin{cases} 1/x^\beta & 0 < x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$



$$\int f_\alpha \overset{\text{(MCT)}}{=} \lim_{n\to\infty} \int_1^n x^{-\alpha}\,dx \overset{\substack{\text{(FTC)} \\ +(L=R)}}{=} \lim_{n\to\infty} \left.\frac{x^{1-\alpha}}{1-\alpha}\right|_1^n \quad (\alpha \neq 1)$$

$$= \lim_{n\to\infty} \frac{n^{1-\alpha} - 1}{1-\alpha}$$

$$\int \chi_{[1,n]} f_\alpha$$

$$= \begin{cases} \frac{1}{\alpha - 1} & \alpha > 1 \\ \infty & \text{otherwise } (\alpha < 1) \end{cases} \quad (\alpha \leq 1)$$

Exercise: $\int g_\beta = \begin{cases} \frac{1}{1-\beta} & \beta < 1 \\ \infty & \beta \geq 1 \end{cases}$

When is $f_\alpha \in L^p$? $\quad p\alpha > 1$

$x^{-\alpha p}$ $\quad p > \frac{1}{\alpha} \quad \alpha > \frac{1}{p}$

Raising p allows for less decay at ∞.

When is $g_\beta \in L^p$? $\quad p\beta < 1, \quad \beta < \frac{1}{p}$

Raising p allows for less blowup at points.

$L_\infty = \{ f, \text{ measurable}, \exists M \text{ such that } |f| \le M \text{ a.e.} \}$

(equivalence classes)

$$\|f\|_\infty = \inf \{ M : |f| \le M \text{ a.e.} \}$$

No decay at ∞ required

No blowups at points is allowed

Are L^p spaces linear? Is $\|\cdot\|_p$ a norm?

$1 \le p < \infty$. $\quad \underline{\underline{f, g \in L^p}}$ $\quad cf \in L^p \ \forall \ c \in \mathbb{R}$.

$$\left(\left(\int |cf|^p \right)^{1/p} = \left(|c|^p \int |f|^p \right)^{1/p} \right.$$

$$= |c| \, \|f\|_p \leftarrow$$

$$|f+g| \le |f| + |g| \le 2\max(|f|, |g|)$$

$$|f+g|^p \le 2^p \max(|f|^p, |g|^p)$$

$$\le 2^p (|f|^p + |g|^p)$$

(L^p is a linear space)

$$\int |f+g|^p \le 2^p \left(\int |f|^p + \int |g|^p \right) < \infty$$

$$\|f+g\|_p \le 2 \left(\|f\|_p^p + \|g\|_p^p \right)^{1/p}$$

$$\|f\|_p = 0 \Rightarrow \int |f|^p = 0 \Rightarrow |f|^p = 0 \text{ a.e.}$$

$$\Rightarrow f = 0 \text{ a.e.}$$

We just need the triangle inequality.

$0 < \frac{1}{p} < 1$

Recall $\quad \alpha^{1/p} \beta^{1/q} \le \frac{1}{p}\alpha + \frac{1}{q}\beta \quad$ if $\quad 1 < p < \infty$

$\frac{1}{p} + \frac{1}{q} = 1$

AM-GM inequality

$$ab \le \frac{1}{p}a^p + \frac{1}{q}b^q$$

$\longrightarrow$ Young's Ineq.

Hölder's Inequality.

Suppose $f \in L^p$ and $g \in L^q$, $\frac{1}{p} + \frac{1}{q} = 1$, $1 < p < \infty$.

Then $|f||g| \in L^1$ and $\int |fg| \le \|f\|_p \|g\|_q$

Pf: Suppose first that $\|f\|_p = 1$ and $\|g\|_q = 1$.

Then $\int |fg| \le \int \left(\frac{1}{p}|f|^p + \frac{1}{q}|g|^q \right)$ (Young's Ineq)

$$= \frac{1}{p}\int |f|^p + \frac{1}{q}\int |g|^q$$

$$= \frac{1}{p} + \frac{1}{q}$$

$$= 1$$

$$= \|f\|_p \|g\|_q$$

If $f = 0$ a.e. or $g = 0$ a.e. the ~~e~~equality is trivial.

Now consider the case $\|f\|_p \neq 0$ and $\|g\|_q \neq 0$.

Let $\tilde{f} = f / \|f\|_p$ and $\tilde{g} = g / \|g\|_q$.

Then $\|\tilde{f}\|_p = 1 = \|\tilde{g}\|_q$.

Hence $\int |\tilde{f}\tilde{g}| \leq 1$.

That is $\int \frac{|fg|}{\|f\|_p \|g\|_q} \leq 1$.

$\square$

Lemma If $f \in L^p$ then $|f|^{p-1} \in L^q$ $\left(\frac{1}{p} + \frac{1}{q} = 1 \right.$
$\left. 1 < p < \infty \right)$

and $\| \, |f|^{p-1} \|_q = \|f\|_p^{p-1}$.