$$F(x) = \int_{-\infty}^{x} f$$

$$= \int \chi_{(-\infty, x]} f$$

$$F(x) - F(y) \qquad x > y$$

$$= \int \chi_{(y, x]} f$$

$f \geq 0 \qquad \int f < \infty$

$$\int f = \lim_{n \to \infty} \int_{E_n} f$$

$$E_n = \{ f \leq n \}$$

$f, g \geq 0 \qquad f = g$ a.e.

$$E = \{f = g\}$$

$\int$

$$f = \chi_E f + \chi_{E^c} f$$

f, g mens

≥ 0

$$\int f = \int (\chi_E f + \chi_{E^c} f)$$

$f = g$ a.e.

$$= \int \chi_E f + \int \underbrace{\chi_{E^c} f}_{= 0 \text{ a.e.}}$$

$\Rightarrow \int f = \int g$

$$= \int_E f + 0$$

$$\int g = \int_E g = \int_E f = \int f$$

$f \geq 0$, $R[a,b]$

$\exists$

$\psi \in Step[a,b]$

$\psi \geq f$

$$\int_a^b f = \inf \{ \int_a^b \psi : \psi \in Step[a,b], \psi \geq f \}$$

$\forall \varepsilon > 0$ $\exists$ $\psi \in$

$\psi_n \geq f$

$$\int_a^b \psi_n \leq \int_a^b f + \frac{1}{n}$$

$\psi_1^* = \psi_1$

$\psi_2^* = \min(\psi_1, \psi_2)$

$$\int_a^b \psi_2^{**} \leq \int_a^b f + \frac{1}{2}$$

$L^1([a,b]) \supseteq \underline{\underline{R[a,b]}} \supseteq C[a,b]$

$C[a,b]$ is <u>dense</u> in $L^1([a,b])$

continuity of integral

$f \to \int_a^b f$

$\underbrace{L^1(\mathbb{R})}$ $\underbrace{\mathcal{L}^1(\mathbb{R})}_{\text{functions}}$

f

E is measurable

$L^1(E)$ equivalence classes of integrable functions on E

$$\int_E |f| < \infty$$

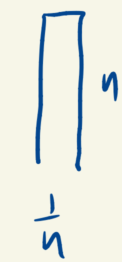
Dominated Convergence Theorem

Suppose (f_n) is a sequence in L_1 and $f_n \to f$ pointwise a.e. Suppose also there exists $g \in L^1$ with $|f_n| \le g$ a.e. for all n. Then $f \in L_1$ and

$f_n \le g$ a.e.

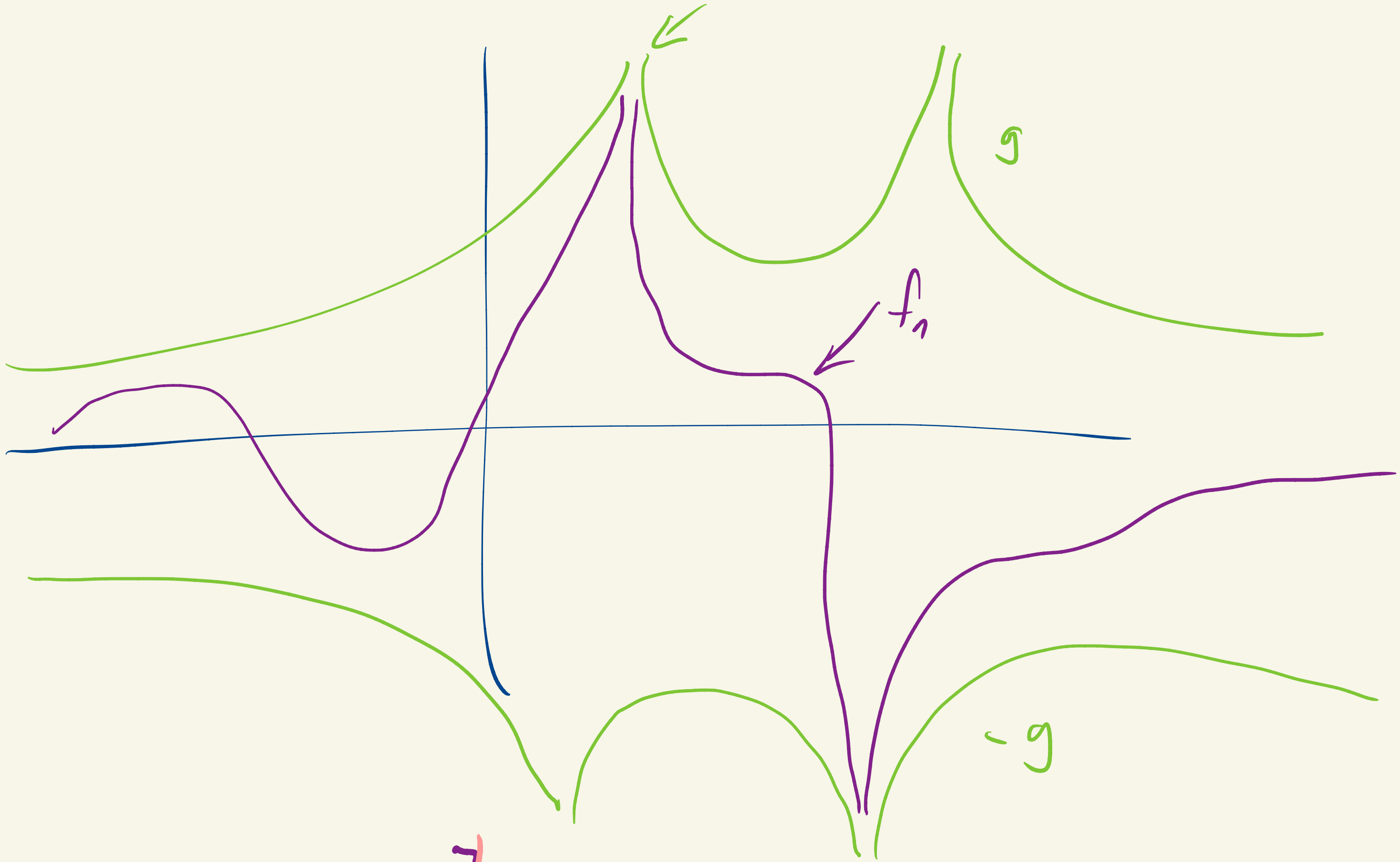
$$\int f_n \to \int f$$

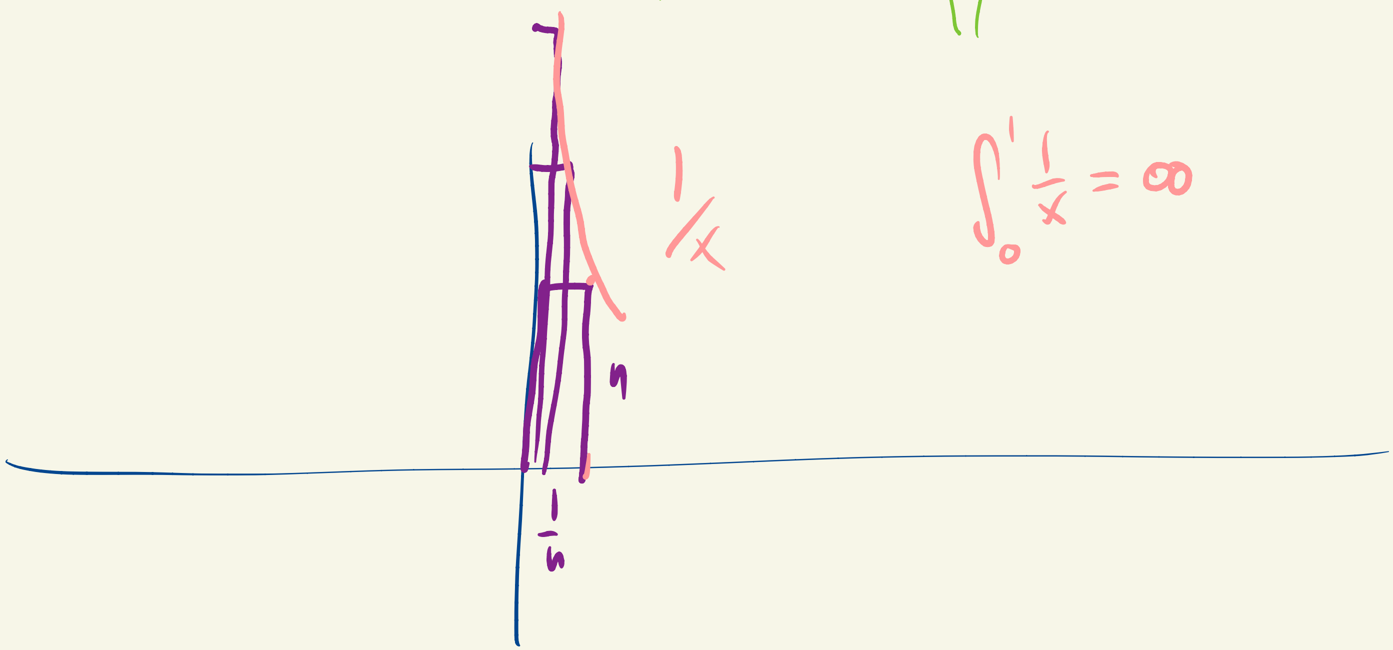
f_n



$\int f_n = 1$

$f_n \to 0$





$\int_0^1 \frac{1}{x} = \infty$

Pf: Upon modification on countably many sets of measure 0 we can assume $f_n \to f$ p.w., that each f_n is finite everywhere, and $|f_n| \le g$ everywhere for all n.

f is finite everywhere, g is finite everywhere

Note that since $|f_n(x)| \le g(x)$ everywhere, $|f(x)| \le g(x)$ everywhere. So $\int |f| \le \int g$. So f is integrable.

$-g \le f_n \le g$

$0 \le g + f_n \le 2g$

Now consider $g + f_n \geq 0$.

Since $g + f_n \to g + f$ pointwise.

From Fatou's Lemma

$$\int g + \int f = \int (g+f) \le \liminf \int (g+f_n) = \int g + \liminf \int f_n$$

Hence

$$\int f \le \liminf \int f_n.$$

On the other hand each $g - f_n \ge 0$ and $g - f_n \to g - f$. p.w.

So the same argument implies

$$\int -f \le \liminf \int -f_n.$$

That is

$$\int f \ge -\liminf\left(-\int f_n\right) = \limsup \int f_n.$$

That is

$$\liminf \int f_n \ge \int f \ge \limsup \int f_n.$$

But $\limsup \int f_n \geq \liminf \int f_n$. Hence $\lim \int f_n = \int f$.

Cor: Under the same hypotheses as above, $f_n \to f$ in L_1.

Pf: $|f_n - f| \leq 2g \in L^1$

$|f_n - f| \to 0$ pw a.e.

So $\int |f_n - f| \to \int 0 = 0$.

That is, $\|f_n - f\|_1 \to 0$.

Cor: If $f \in L_1$, $F(x) = \int_{-\infty}^{x} f$ is continuous.

Pf: Suppose $x_n \to x$.

$$\chi_{(-\infty,-1/n)} \to \chi_{(-\infty,0)}$$

Then $\chi_{(-\infty,x_n]} f \to \chi_{(-\infty,x]} f$ p.w. a.e.

Moreover $|\chi_{(-\infty,x_n]} f| \le |f|$ for all n.

$\hookrightarrow \in L^1$.

So $\displaystyle\int_{-\infty}^{x_n} f \to \int_{-\infty}^{x} f$.

Cor: If (f_n) is a sequence of functions in L_1 and $\sum_{n=1}^{\infty} \|f_n\|_1 < \infty$ then $\sum_{n=1}^{\infty} f_n$ converges in L_1.

(I.e., absolutely convergent series in L_1 converge and hence L_1 is complete).

Pf: Let $g = \sum_{n=1}^{\infty} |f_n|$. By the MCT,

$$\int g = \sum_{n=1}^{\infty} \int |f_n| < \infty.$$

So $g \in L_1$ and is finite a.e.

At each x where g is finite,

$$\sum_{n=1}^{\infty} f_n(x)$$ converges (because it converges absolutely).

Let the limit be $f(x)$ which we extend by 0 to

that when $g = \infty$.

Let $s_n = \sum_{k=1}^{n} f_k$. Then $|s_n| \leq g$.

Since $s_n \to f$ p.w. a.e.

Hence $$\int f = \lim_{n\to\infty} \int s_n = \lim_{n\to\infty} \int \sum_{k=1}^{n} f_k$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \int f_k$$

$$= \sum_{k=1}^{\infty} \int f_k .$$

Moreover, each $|f - s_n| \leq \underbrace{2g}_{\in L^1}$, and $|f - s_n| \to 0$ p.w. a.e.

So $\int |f - s_n| \to 0$. So $s_n \to f$ in L_1.

Moreover, by the cor above, $s_n \to f$ in L_1.

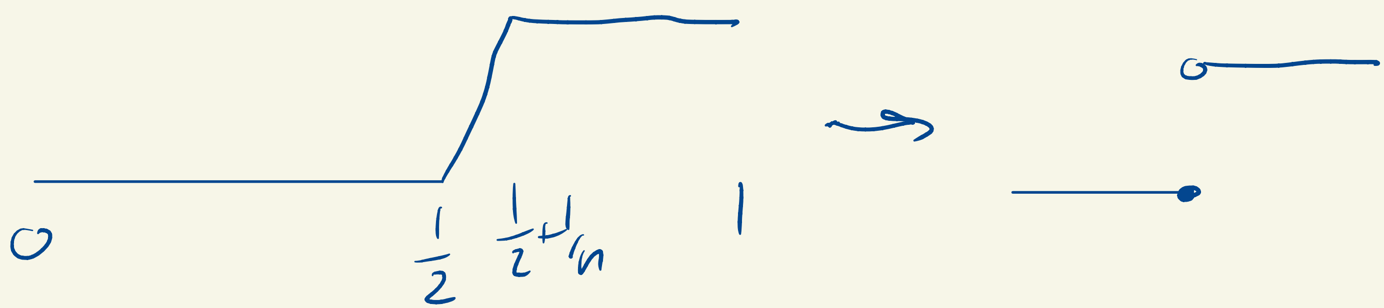
That is $\sum_{k=1}^{\infty} f_k = f$ in L_1 $\square$

$$\lim_{n\to\infty} \sum_{k=1}^{n} f_k = f$$

Λ

$C[a,b] \subseteq L_1[a,b]$

$C[a,b]$ with the L_1 norm is not complete.



What is $\overline{C[a,b]} = L^1[a,b]$

Thm: Let f be integrable on $\mathbb{R}$.

a) For every $\varepsilon > 0$ there is an integrable simple function φ with $\int |f - \varphi| < \varepsilon$.

b) For every $\varepsilon > 0$ there is a continuous function g that vanishes outside a bounded interval such that

$$\int |f - g| < \varepsilon.$$

Let $\varepsilon > 0$.

Pf: Consider $f_n = \chi_{\{|x| > n\}} f$.

Then $|f_n| \leq |f| \in L^1$ and $f_n \to 0$ pointwise.

Hence $\int |f_n| \to 0$. (DCT)

In particular we can find a bounded interval I

such that $\int_{I^c} |f| < \frac{\varepsilon}{2}$.

From the Basic construction we can find integrable

simple functions φ_n with $|\varphi_n| \leq \chi_I f$

with $\varphi_n \to \chi_I f$ pointwise.

Hence $\varphi_n \to \chi_I f$ in L^1.

So for n large enough, $\int |\varphi_n - \chi_I f| < \frac{\varepsilon}{2}$.

So $$\int |\varphi_n - f| = \int_{I^c} |\varphi_n - f| + \int_I |\varphi_n - f|$$

$$= \int_{I^c} |f| + \int |\varphi_n - \chi_I f|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$\square$

Pf of b)

Given $f \in L^1$ find an integrable simple function φ with bounded support (on some interval I, say) such that

$\| f - \varphi \|_1 < \varepsilon$.