$$\|f\|_1 = \underline{\underline{\int |f|}}$$

We'll fix up L_1.

Recall: 1) if $f \geq 0$ and $\int f < \infty$ then f is finite a.e.

2) if $f \geq 0$ then $\int f = 0 \underline{\Leftrightarrow} f = 0$ a.e.

Observations

a) If $f \in L^1$ then f is finite a.e.

$\int f_+ < \infty \quad \int f_- < \infty \Rightarrow f_+ < \infty$ and $f_- < \infty$ a.e.

$\Rightarrow f$ is finite a.e.

b) If $f = 0$ a.e then $f \in L^1$ and $\int f = 0$

(it's measurable!)

$|f| \geq 0 \quad |f| = 0$ a.e. $\quad \int |f| = 0 \Rightarrow f \in L^1$

$f_+ = 0$ a.e. $\quad \left.\begin{array}{l} \int f_+ = 0 \\ \int f_- = 0 \end{array}\right] \Rightarrow \int f = 0$

c) If $g \in L^1$ and if $f = g$ a.e. then $f \in L^1$

and $\int f = \int g$ (measurability is easy!)

$\{f_+ \neq g_+\} \subseteq \underbrace{\{f \neq g\}}_{\text{null}}$

$f_+, g_+ \geq 0$ and $f_+ = g_+$ a.e. So $\int f_+ = \int g_+ < \infty$

Similarly $\int f_- = \int g_- < \infty$.

$\Rightarrow$ $f \in L_1$. Moreover

$$\int f = \int f_+ - \int f_- = \int g_+ - \int g_- = \int g.$$

d) If $g \in L^1$ and if $f = g$ a.e. (so $f \in L^1$)

then for all measurable sets E

$$\int_E g = \int_E f$$

$$\int |\chi_E g| \leq \int |g| < \infty$$

$$\int_E g = \int \chi_E g \qquad \chi_E g = \chi_E f \quad \text{a.e.}$$

$$\Rightarrow \int \chi_E g = \int \chi_E f$$

$$E \longmapsto \int_E f$$

e) If $h \in L^1$ and $\int_E h = 0$ for all measurable sets E then $h = 0$ a.e.

Let $E = \{h \geq 0\}$. $\int_E h = 0$.

$$\int \chi_E h = 0$$

Claim $\chi_E h = h_+$.

$\Rightarrow \int h_+ = 0$ So $h_+ = 0$ a.e.

Similarly $h_- = 0$. a.e.

But then $h = h_+ - h_- \geq 0$ a.e.

f) If $f, g \in L^1$ and $\int_E f = \int_E g$ for all

measurable sets E then $f = g$ a.e.

Let $F = \{ |f| < \infty \}$

Let $G = \{ |g| < \infty \}$

Let $\tilde{f} = \chi_F f$, $\tilde{g} = \chi_G g$.

Then $\tilde{f}$ and $\tilde{g}$ are finite everywhere and

$\tilde{f} = f$ a.e. and $\tilde{g} = g$ a.e.

Let $h = \tilde{f} - \tilde{g}$. (Well defined!) (in L^1)

Then for any measurable set E

$$\int_E h = \int_E (\tilde{f} - \tilde{g}) = \int_E \tilde{f} - \int_E \tilde{g}$$

$$= \int_E f - \int_E g$$

$$= 0$$

So $h = 0$ a.e. So $\tilde{f} = \tilde{g}$ a.e.

So $f = g$ a.e.

g) If $f \in L^1$ and $f \geq 0$ a.e. then $\int f \geq 0$.

$$f_- = 0 \quad \text{a.e.} \qquad (-f) \vee 0$$

$$\int f_- = 0$$

$$\int f = \int f_+ - \int f_- = \int f_+ \geq 0.$$

h) If $f, g \in L^1$ and $f \leq g$ a.e.

$$\int f \leq \int g.$$

$$\tilde{g} - \tilde{f} \geq 0 \quad \text{a.e.}$$

$$\int (\tilde{g} - \tilde{f}) \geq 0$$

$$\int \tilde{g} - \int \tilde{f} \geq 0$$

$$\int \tilde{g} \geq \int \tilde{f}$$

$$\int g \geq \int f$$

Official

Equivalence relation on integrable functions. $\left(\begin{array}{c} f \text{ meas} \\ \int |f| < \infty \end{array} \right)$

$$f \sim g \quad \text{if} \quad f = g \text{ a.e.}$$

$$L^1 = \{ [f] : f \text{ is meas and integrable} \}$$

We define $\int [f] = \int f$ (well defined?)

$$[f] + [g] = [\underbrace{\tilde{f} + \tilde{g}}_{\hookrightarrow \text{ integrable}}] \qquad \text{(well defined)}$$

$$\int ([f] + [g]) = \int [\tilde{f} + \tilde{g}] = \int (\tilde{f} + \tilde{g})$$

$$= \int \tilde{f} + \int \tilde{g}$$

$$= \int [\tilde{f}] + \int [\tilde{g}]$$

$$= \int [f] + \int [g]$$

$c[f] = [cf]$ (Exercise: this is well defined

and $\int c[f] = c \int [f]$

L^1 is a vector space and

$[f] \to \int [f]$ is linear on it.

$$\|[f]\|_1 = \int |f| \qquad \text{(Well defined!)}$$

Is this a norm?

$\|[f]\|_1 \geq 0$? Yep.

$\|[0]\|_1 = 0$? Yep.

$\|[f]\|_1 = 0 \Rightarrow [f] = [0]$?

$\hookrightarrow \int |f| = 0 \Rightarrow |f| = 0$ a.e.

$\Rightarrow f = 0$ a.e

$\Rightarrow [f] = [0]$

$$\|c[f]\|_1 = \|[cf]\|_1 = \int |cf|$$
$$= \int |c|\,|f|$$
$$= |c| \int |f|$$
$$= |c|\, \|[f]\|_1$$

$$\|[f] + [g]\|_1 = \|[\tilde{f} + \tilde{g}]\|_1$$
$$= \int |\tilde{f} + \tilde{g}|$$
$$\leq \int \left(|\tilde{f}| + |\tilde{g}|\right)$$
$$= \int |\tilde{f}| + \int |\tilde{g}|$$

$$= \int |f| + \int |g|$$

$$= \|[f]\|_1 + \|[g]\|_1$$

This is a norm on L^1.

Is $[f] \to \int [f]$ continuous?

$$\left| \int [f] \right| = \left| \int \tilde{f} \right|$$

$$= \left| \int \tilde{f}_+ - \int \tilde{f}_- \right|$$

$$\leq \left| \int \tilde{f}_+ \right| + \left| \int \tilde{f}_- \right|$$

$$= \int \hat{f}_+ + \int \hat{f}_-$$

$$= \int \tilde{f}_+ + \tilde{f}_-$$

$$= \int |\tilde{f}| = \int |f|$$

$$= \|[f]\|_1$$

$$\left| \int [f] \right| \leq \|[f]\|_1$$

$c = 1$

continuity of integration

$f \in L^1$

$\uparrow$

$f_n \geq 0$ $\quad$ $f_n \to f$ $\quad$ $\int f \leq \liminf \int f_n$

$f_n \geq 0$ a.e. $\quad$ $f_n \to f$ pw a.e.

$\downarrow$ $\quad$ $\tilde{f}_n \to f$ pw a.e.

$\tilde{f}_n \geq 0$ $\quad$ $\tilde{\tilde{f}}_n \to \tilde{f}$ pw.

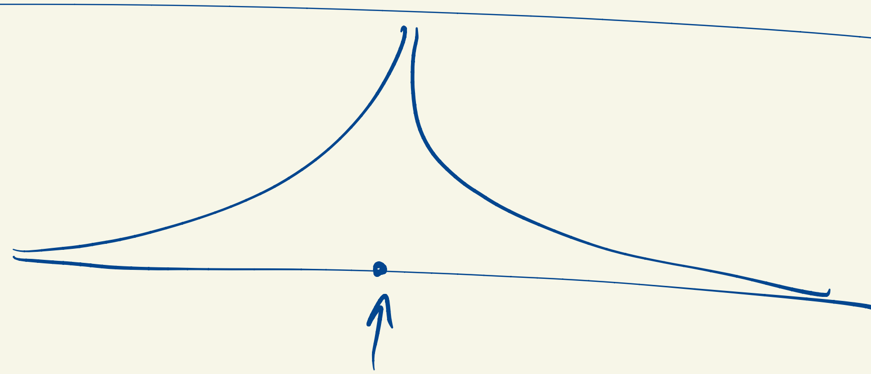
$\tilde{\tilde{f}}_n \geq 0$ $\quad$ $\tilde{\tilde{f}}_n = f_n$ a.e.

$\tilde{f} = f$ a.e.

$$\int \tilde{f} \le \liminf \int \tilde{f}_n$$

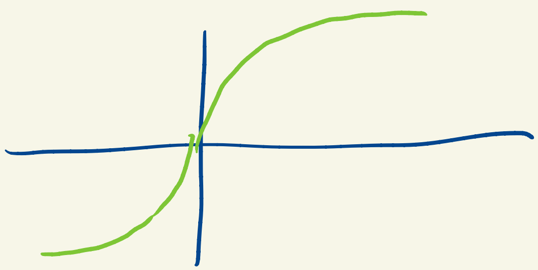
$$\int f \le \liminf \int f_n$$

MCT



$f_n \geq 0$ a.e.

$f_n \nearrow f$ pw a.e.



$$\lim_{n\to\infty} \int f_n = \int f$$