If $f_n \to f$

$$\int f \leq \liminf_{n\to\infty} \int f_n$$ (Fatou's Lemma)

MCT $f_n \geq 0$ f_n increasing pointwise (to f)
(measurable)

$$\int f_n \to \int f$$ (continuity from below)

$f_n \geq 0$ measurable.

$$f = \sum_{n=1}^{\infty} f_n \qquad \int f = \sum_{n=1}^{\infty} \int f_n \ ?$$

$$s_n = \sum_{j=1}^{n} f_j \qquad s_n \nearrow f \qquad (f_j \geq 0)$$

$$\int s_n \to \int f \qquad (\text{by MCT})$$

$$\underbrace{\int (f+g) = \int f + \int g}_{f,g \geq 0}$$

$$\downarrow$$

$$\int s_n = \int \sum_{j=1}^{n} f_j = \sum_{j=1}^{n} \int f_j$$

$$\sum_{j=1}^{n} \int f_j \to \int f$$

$$\sum_{j=1}^{\infty} \int f_j = \int f$$

Monotone decreasing? $\quad f_n \geq 0$, decreasing. $\quad f_n \searrow f$

(measurable)

$$\int f_n \xrightarrow{?} \int f$$

$$f_n = \frac{1}{n} \quad \forall n.$$

$$\int f_n = \infty$$

$$f_n \to 0 \qquad \int 0 = 0$$

No, but

if $\int f_1 < \infty$ then yes!

$$f_1 = \overbrace{(f_1 - f_n)}^{\infty - \infty = 0} + f_n$$

$$\int f_1 = \int (f_1 - f_n) + \int f_n$$

$$\underbrace{\underbrace{f_1 - f_n}_{\geq 0} \nearrow f_1 - f}_{\int f_1 - f_n \to \int f_1 - f}$$

$$\int f_n = \int f_1 - \int (f_1 - f_n)$$

$\int f_1 < \infty$

$$\int f_n \to \int f_1 - \int (f_1 - f) = \int f$$

$$E_n \searrow E \qquad \int_{E_n} f \to \int_E f \qquad \text{If } \int_{E_1} f < \infty$$

$$\chi_{E_n} f \searrow \chi_E f$$

(Basic)

Fatou's Lemma: Suppose (f_n) are measurable and non-negative.

If $f_n \to f$ pointwise then

$$\int f \leq \liminf_{n \to \infty} \int f_n .$$

(You can loose area along the way but you can't gain it)

Pf: Let $g_n = \inf_{k \geq n} f_k$. Note $0 \leq g_n \leq f_n$ for all n.

Moreover, the g_n's are monotone increasing to

$$\lim g_n = \lim_{n\to\infty} \inf_{k \geq n} f_k = \boxed{\liminf_{n\to\infty} f_n} = \lim_{n\to\infty} f_n = f.$$

So by the MCT, $\int g_n \to \int f$.

Since $g_n \leq f_n$ for all n, $\int g_n \leq \int f_n$ for all n and

$$\lim_{n\to\infty} \int g_n = \liminf_{n\to\infty} \int g_n \leq \liminf_{n\to\infty} \int f_n.$$

That is, $\int f \leq \liminf_{n\to\infty} \int f_n$.

More generally: $\int \liminf f_n \leq \liminf \int f_n$

$f_n \geq 0$

If you already know Fatou's Lemma, the MCT is a consequence.

$$f_n \nearrow f \qquad \int f_n \leq \int f$$

$$\lim_{n\to\infty} \int f_n \leq \int f$$

But Fatou's Lemma implies

$$\int f \leq \liminf \int f_n = \lim \int f_n.$$

Integration of arbitrary functions

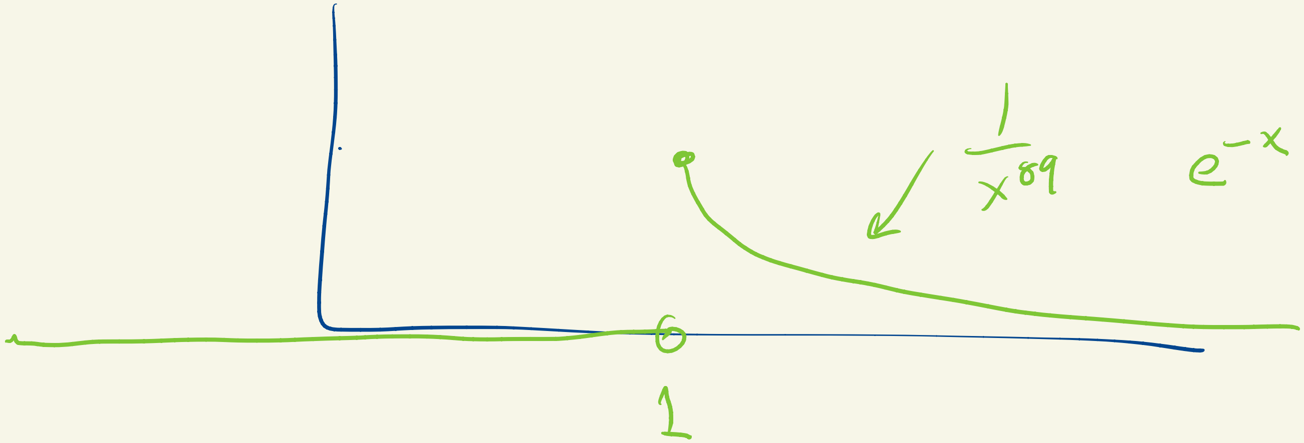
f measurable

$$f = f_+ - f_- \qquad \left.\begin{aligned} f_+ &= f \vee 0 \\ f_- &= (-f) \vee 0 \end{aligned}\right] \begin{array}{l}\text{measurable} \\ \geq 0\end{array}$$

$\int f := \int f_+ - \int f_-$ so long as one of the summands is finite

$(\infty - \infty)$

If $\int f_+$ and $\int f_-$ are both finite we say f is <u>integrable</u>. (Exercise: f is integrable iff $\int |f| < \infty$)



Provisionally: $L^1(\mathbb{R})$ (L^1) is the set of integrable functions.

We'd like to show that L^1 is a vector space,

$f \longmapsto \underline{\int |f|}$ is a norm on L^1 and

$f \longmapsto \int f$ is a linear function on L^1.

$f = \chi_{\mathbb{Q}}$ $\quad \int |f| = 0$ but $f \neq 0$.

is $f+g$ even well defined? $\quad \infty - \infty$

Claim: If $f, g \in L^1$ and are finite everywhere then

$$f+g \in L^1 \quad \text{and} \quad \int (f+g) = \int f + \int g$$

$|f+g| \leq |f| + |g|$ so by monotonicity $f+g \in L^1$.

$$\left(\int |f+g| \leq \int (|f| + |g|) = \int |f| + \int |g| < \infty \right)$$

$$f+g = (f+g)_+ - (f+g)_-$$

$$f+g = f_+ - f_- + g_+ - g_-$$

$$(f+g)_+ + f_- + g_- = (f+g)_- + f_+ + g_+$$

$$\int \left[(f+g)_+ + f_- + g_- \right] = \int \left[(f+g)_- + f_+ + g_+ \right]$$

$$\int (f+g)_+ + \int f_- + \int g_- = \int (f+g)_- + \int f_+ + \int g_+$$

$$\int (f+g)_+ - \int (f+g)_- = \int f_+ - \int f_- + \int g_+ - \int g_-$$

$$\int (f+g) = \int f + \int g$$

:)

Claim: If $f \in L^1$ and $c \in \mathbb{R}$ then $cf \in L^1$

and $\int cf = c \int f$

$$|cf| = |c|\,|f| \qquad \int |c|\,|f| = |c| \int |f| < \infty$$

$$\Rightarrow \quad cf \in L^1$$

If $\quad c \geq 0 \qquad (cf)_+ = c f_+$

$$(cf)_- = c f_-$$

$$\underbrace{\int (cf)_+ - \int (cf)_-}_{\int cf} = \int c f_+ - \int c f_-$$

$$= c \int f_+ - c \int f_-$$

$$= c \left(\int f_+ - \int f_- \right)$$

$$= c \int f$$

If $c = -1$ $\quad (cf)_+ = (-f)_+ = f_-$

$$(cf)_- = (-f)_- = f_+$$

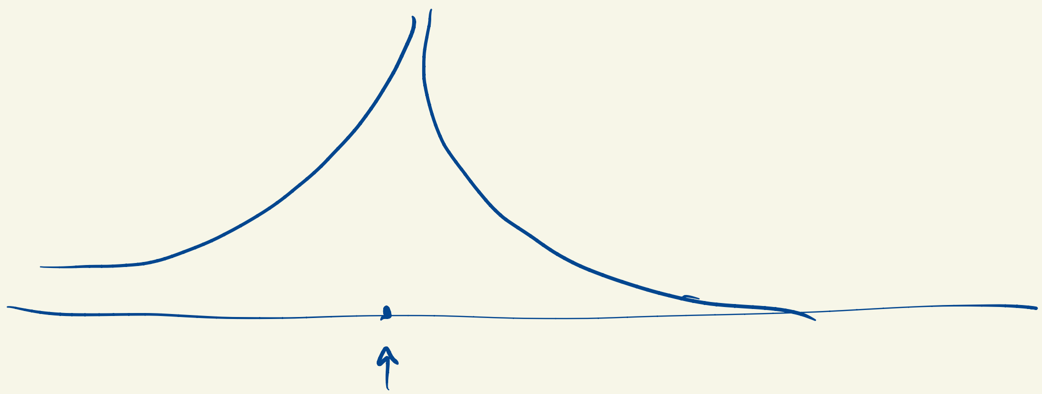
$$\int cf = \int (cf)_+ - \int (cf)_-$$

$$= \int f_- - \int f_+$$

$$= -\left(\int f_+ - \int f_- \right)$$

$$= - \int f = c \int f$$

The finite everywhere elements of L^1 form a vector space and $f \longmapsto \int f$ is a linear map on it.



$$\frac{1}{x^{1/4}} \in H^1([-1,1])$$