Monotone Convergence Theorem

$f_n : \mathbb{R} \to [0, \infty]$

Let (f_n) be a sequence of increasing non-negative measurable functions, so $f_n \le f_{n+1}$ for all n.

Define $f = \lim f_n$. Then $\lim_{n \to \infty} \int f_n = \int f$.

Pf (MCT)

Since each $f_n \leq f$ for all n, $\int f_n \leq \int f$ for all n.

Hence $\lim_{n\to\infty} \int f_n \leq \int f$.

So it suffices to show $\lim_{n\to\infty} \int f_n \geq \int f$.

$L \geq \alpha S$

$\forall\ 0<\alpha<1$

Recall $\int f = \sup\{\int \varphi : \varphi$ is simple, integrable, $0 \leq \varphi \leq f\}$.

Suppose φ is simple, integrable and $0 \leq \varphi \leq f$. It suffice to show that for all $0<\alpha<1$ that $\lim_{n\to\infty} \int f_n \geq \alpha \int \varphi$.

Let $0<\alpha<1$. Consider $\alpha\varphi$. Then $0 \leq \alpha\varphi < f$ everywhere.

Let $E_n = \{ f_n \geq \alpha \varphi \}$. Observe that each $E_{n+1} \supseteq E_n$ and $\bigcup E_n = \mathbb{R}$. $\left(f_n(x) \to f(x) > \alpha \varphi(x) \right)$

For each n,
$$\int f_n \geq \int_{E_n} f_n \geq \int_{E_n} \alpha \varphi .$$

So
$$\lim_{n\to\infty} \int f_n \geq \lim_{n\to\infty} \int_{E_n} \alpha \varphi = \int_{\mathbb{R}} \alpha \varphi .$$

$\square$

Cor: Suppose $f \geq 0$ and measurable. If the sets E_n are increasing ($E_{n+1} \supseteq E_n$) and measurable then
$$\lim_{n\to\infty} \int_{E_n} f = \int_E f \quad \text{where } E = \bigcup E_n .$$

$$\underbrace{\chi_{E_n} f}_{\geq 0} \nearrow \chi_E f$$

$$\int (f+g) = \int f + \int g$$

Lemma: If f is non-negative and measurable then is a sequence of non-negative integrable simple functions φ_n with $0 \leq \varphi_n \leq f$ and $\varphi_n \nearrow f$.

Pf: From the basic construction let ψ_n be an increasing sequence of simple functions with

$$0 \le \psi_n \le f \quad \text{and} \quad \psi_n \nearrow f \text{ pointwise.}$$

Let $\varphi_n = \chi_{[-n,n]} \psi_n$.

Note: in the above $\int f = \lim \int \varphi_n$.

Prop: If $f, g \ge 0$ and measurable then

$$\int (f+g) = \int f + \int g.$$

Pf: Let f_n and g_n be increasing sequences of non-negative simple functions converging pointwise to f and g respectively. So $\int f = \lim_n \int f_n$ and similarly for g.

Now
$$\lim_{n\to\infty} \int (f_n + g_n) = \lim_{n\to\infty} \left(\int f_n + \int g_n \right)$$
$$= \lim_{n\to\infty} \int f_n + \lim_{n\to\infty} \int g_n$$
$$= \int f + \int g.$$

On the other hand, the sequence $f_n + g_n$ increases to $f+g$. So the MCT implies

$$\lim_{n\to\infty} \int (f_n + g_n) = \int f+g.$$

(f_n) $\quad f_n \geq 0$ measurable

$$f_n \to f$$

$\int f_n \to \int f \leftarrow$ MCT

1) $f_n = \chi_{[n,\infty)}$ $\quad \int f_n = \infty$ but $\int f_n \to \int 0 = 0$

$f_n \to 0$

2) $f_n = \chi_{[n,n+1]}$ $\quad \int f_n = 1$ $\quad \int f_n \not\to 0$

$f_n \to 0$

3) $f_n = \frac{1}{n}\chi_{[0,n]}$ $\quad \int f_n = 1$ $\quad \int f_n \not\to 0$

$f_n \rightrightarrows 0$

4) $f_n = n\chi_{(0,1/n]}$ $\quad \int f_1 = 1$ $\quad \int f_n \not\to 0$

$f_n \to 0$

5) $f_n = \begin{cases} \chi_{[n, n+1]} & n \text{ is odd} \\ \chi_{[n, n+3]} & n \text{ is even} \end{cases}$

$f_n \to 0 \qquad \int f_n = 2 + (-1)^n$

1) - 4) $\int f \leq \lim\limits_{n\to\infty} \int f_n$

5) $\int f \leq \liminf\limits_{n\to\infty} \int f_n$

always holds

If $f_n \to f$

$$\int f \leq \liminf_{n\to\infty} \int f_n$$ (Fatou's Lemma)

MCT $f_n \geq 0$ f_n increasing pointwise (to f)
(measurable)

$$\int f_n \longrightarrow \int f$$ (continuity from below)

$f_n \geq 0$ measurable

$$f = \sum_{n=1}^{\infty} f_n \qquad \int f = \sum_{n=1}^{\infty} \int f_n \ ?$$