Integration

A simple function $\varphi : \mathbb{R} \to \mathbb{R}$ is *integrable* if $m(\{\varphi \neq 0\}) < \infty$.

If $\varphi = \sum_{k=1}^{n} a_k \chi_{E_k}$ in *standard form* then

distinct disjoint

$$I(\varphi) = \sum_{k=1}^{n} a_k \, m(E_k)$$

(if $a_k \neq 0$ then $m(E_k) < \infty$ and we interpret $0 \cdot \infty = 0$)

I'd like to show

$$I(\varphi+\psi) = I(\varphi) + I(\psi)$$] tedious

$$I(c\varphi) = c\, I(\varphi)$$] super easy

If $\varphi \geq \psi \implies I(\varphi) \geq I(\psi)$.] pretty easy

Lemma: If φ is simple and integrable and if $\varphi = \sum_{k=1}^{K} b_k \chi_{F_k}$ with the measurable sets F_k disjoint then

$$I(\varphi) = \sum_{k=1}^{K} b_k\, m(F_k)$$

Pf $\quad I(\varphi) = \sum_{a \in \mathbb{R}} a \, m(\{\varphi = a\})$

But $\quad m(\{\varphi = a\}) = m\left(\bigcup_{b_k = a} F_k\right)$

$$= \sum_{b_k = a} m(F_k).$$

So $\quad I(\varphi) = \sum_{a \in \mathbb{R}} a \sum_{b_k = a} m(F_k)$

$$= \sum_{a \in \mathbb{R}} \sum_{b_k = a} b_k \, m(F_k)$$

$$= \sum_{k=1}^{K} b_k \, m(F_k). \qquad \square$$

Prop: If φ and ψ are integrable simple functions then so is $\varphi + \psi$ and

$$I(\varphi+\psi) = I(\varphi) + I(\psi).$$

Pf: It is obvious that $\varphi + \psi$ is simple and integrable.

We represent

$$\varphi = \sum_{i=1}^{n} a_i \chi_{E_i}$$

$$\psi = \sum_{j=1}^{m} b_j \chi_{F_j}$$

in standard form.

Let $A_{ij} = E_i \cap F_j$, so $\bigcup_i A_{ij} = F_j$ and $\bigcup_j A_{ij} = E_i$.

Then $$I(\varphi+\psi) = \sum_{i=1}^{n} \sum_{j=1}^{m} (a_i + b_j)\, m(\underbrace{A_{ij}}_{\text{disjoint!}})$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} a_i\, m(A_{ij}) + \sum_{j=1}^{m} \sum_{i=1}^{n} b_j\, m(A_{ij})$$

$$= \sum_{i=1}^{n} a_i\, m(E_i) + \sum_{j=1}^{m} b_j\, m(F_j)$$

$$= I(\varphi) + I(\psi).$$

$\square$

Cor: If each E_i is measurable with finite measure and

$$\varphi = \sum_{i=1}^{n} a_i \chi_{E_i}$$ then

$$I(\varphi) = \sum_{i=1}^{n} a_i\, m(E_i)$$

If φ is integrable and simple and $\varphi \geq 0$ a.e.,

then $I(\varphi) \geq 0$.

standard form

$$\varphi = \sum_{k=1}^{n} a_k \chi_{E_k} \quad \text{where } m(E_k) = 0 \text{ if } a_k < 0.$$

$$I(\varphi) = \sum a_k m(E_k) \geq 0$$

Cor: If φ and ψ are simple and integrable and

$\varphi \geq \psi$ a.e. then $I(\varphi) \geq I(\psi)$

Pf: $\varphi - \psi \geq 0$ a.e. so $\int(\varphi - \psi) \geq 0$.

But $\int(\varphi - \psi) = \int\varphi - \int\psi$. So $\int\varphi \geq \int\psi$.

Suppose $f : \mathbb{R} \to \mathbb{R}$ is measurable and $f \geq 0$.

$$\int f = \sup \{ I(\varphi) : 0 \leq \varphi \leq f \text{ and } \varphi \text{ is simple+ integrable} \}$$

$\hookrightarrow$ ∞ is a possibility

If $\int f < \infty$ we say f is <u>integrable</u>.

Exercise: If φ is simple and integrable and non-negative then $I(\varphi) = \int \varphi$.

(Monotonicity!)

$\psi \leq \varphi$

$I(\psi) \leq I(\varphi)$

$\int \Rightarrow$ over all of $\mathbb{R}$

Exercise: If $f \geq 0$ and measurable and $\alpha \geq 0$
then $\int \alpha f = \alpha \int f$

Exercise: If $f \geq g \geq 0$ are measurable then

$$\int f \geq \int g$$

Harder: $\int (f+g) = \int f + \int g$ Stay tuned!

If $D \subseteq \mathbb{R}$ is measurable and $f : \mathbb{R} \to [0,\infty]$ is measurable
and non negative

$$\int_D f = \int \chi_D f.$$

If D is measurable and $f: D \to [0,\infty]$ is measurable

$$\int_D f = \int \hat{f} \quad \text{where} \quad \hat{f}(x) = \begin{cases} f(x) & x \in D \\ 0 & x \in D^c \end{cases}$$

If $D \supseteq E$ where E is measurable, $(f: D \to [0,\infty])$

$$\int_E f = \int_D \chi_E f = \int \widehat{\chi_E f} = \int \chi_E \hat{f}$$

e.g. $\int \chi_E = m(E)$

If $m(E) < \infty$ this follows since $\int \chi_E = I(\chi_E)$.

$\varphi_n = \chi_{E \cap [-n,n]}$ $\quad \varphi_n \leq \chi_E$

$$I(\varphi_n) = m(E \cap [-n,n]) \to m(E) = \infty.$$

$$\int \chi_E \geq I(\varphi_n) \quad \forall n,$$

Chebyshev's Inequality

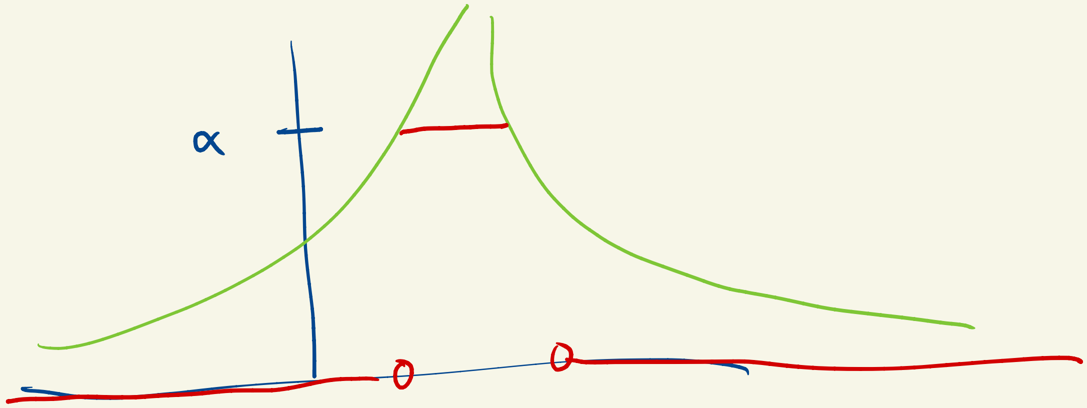
If $f \geq 0$ and measurable, then for all $\alpha \geq 0$

$$\int f \geq \alpha \, m(\{f \geq \alpha\})$$

Exercise: verify this is ok if $m(E_\alpha) = \infty$.

Pf: Observe $f \geq \alpha \chi_{E_\alpha}$ where $E_\alpha = \{f \geq \alpha\}$.

Then $$\int f \geq \boxed{\int \alpha \chi_{E_\alpha} = \alpha \, m(E_\alpha)} = \alpha \, m(\{f \geq \alpha\})$$



Cor: If $f \geq 0$ is measurable and $\int f = 0$

then $f = 0$ a.e.

Pf: Let $E_n = \{ f \geq \frac{1}{n} \}$. Then

$$0 = \int f \geq \frac{1}{n} m(E_n) \quad \text{by Chebyshev's ineq.}$$

So $m(E_n) = 0$ for all n. But $\{f \neq 0\} = \bigcup_n E_n$

is a union of null sets.

Cor: Suppose $f \geq 0$ and measurable and $\int f < \infty$.

Then $\{ f = \infty \}$ is null. That is,

f is finite a.e.

Pf: Let $E_n = \{ f \geq n \}$. Then

$$\int f \geq n\, m(E_n) \quad \text{by Chebyshev's ineq.}$$

So $\quad m(E_n) \leq \frac{1}{n} \int f.$

Hence $\quad m(E_n) \longrightarrow 0.$

Since E_1 has finite measure and since

$E_{n+1} \subseteq E_n$ for each n, continuity from above

implies $m\left(\bigcap E_n\right) = 0$.

But $\bigcap E_n = \{f = \infty\}$.

$f_n \geq 0$ $\quad f_n \to f$ pointwise

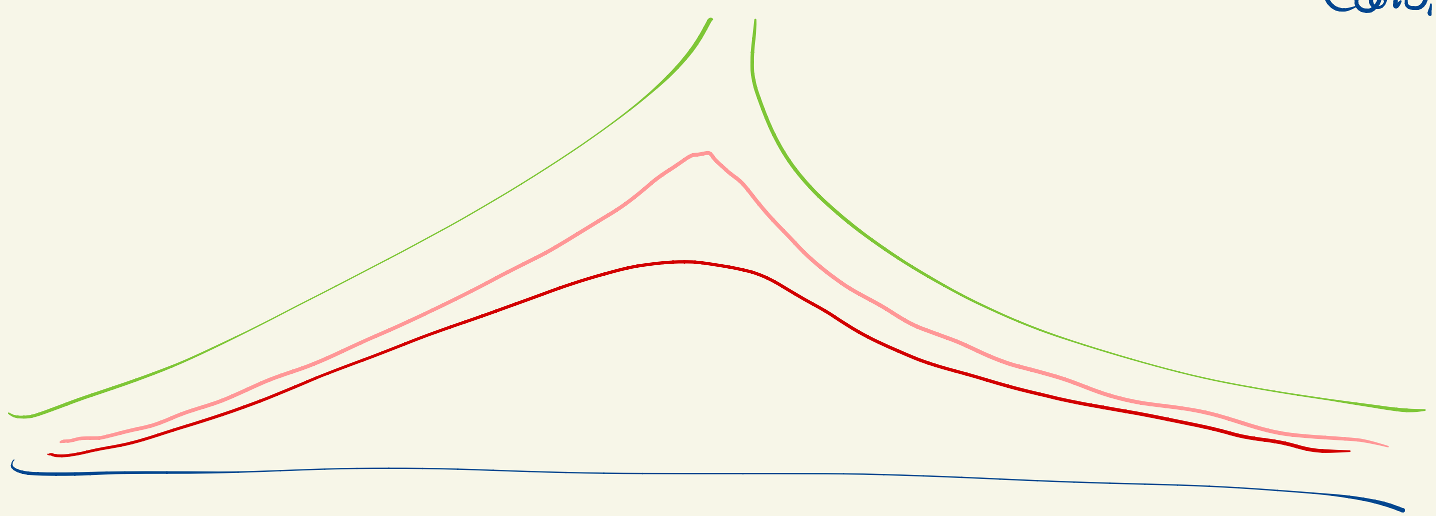
$$\int f_n \to \int f \quad ?$$

No: $\quad f_n = \chi_{[n, n+1]} \qquad \int f_n = 1 \qquad f_n \to 0$

We something extra. This is true if $f_n \nearrow f$.

$f_n \leq f_{n+1}$ $\qquad$ $f_n \to f$ p.w.

Claim $\int f_n \to \int f$ $\qquad$ (Monotone Conv. Thm)



Lemma: Let φ be simple and integrable,

If $E_1 \subseteq E_2 \subseteq \cdots$ and $E = \bigcup E_k$

Then $\int_{E_k} \varphi \longrightarrow \int_E \varphi$

Pf: Let $\varphi = \sum a_k \chi_{F_k}$ in standard form.

So $\int \varphi = \sum_{k=1}^{m} a_k \, m(F_k)$

Then $\int_{E_n} \varphi = \int \chi_{E_n} \varphi = \sum_{k=1}^{m} a_k \, m(E_n \cap F_k)$

$\longrightarrow \sum_{k=1}^{m} a_k \, m(E \cap F_k)$

$$= \int_E \varphi$$

$$\int_{-\infty}^{x} f \rightarrow \geq 0 \text{ meas.}$$

f meas.

$f_+ - f_-$ $\quad f_+, f_- \geq 0$

$$\int f = \underline{\int f_+ - \int f_-}$$