That is, if f is measurable then

$$f^{-1}(B) \in \mathcal{M} \quad \text{for all Borel sets } B.$$

$f : \mathbb{R} \to \mathbb{R}$

If f is continuous, f is measurable

$$f^{-1}(\text{open}) = \text{open}$$

If f is monotone increasing, f is measurable

$$f^{-1}((a,\infty)) \text{ is an interval,}$$

If f is ~~lower~~ upper semicontinuous, f is measurable

If N is null and $f: N \to \mathbb{R}$ then f is measurable.

If $f: D \to \mathbb{R}$ is measurable and if $g: D \to \mathbb{R}$ and

$g = f$ except on a null set then g is measurable

" $g = f$ $\underbrace{\text{almost everywhere}}$ "

a.e. p.p. (except on a null set) $f^{-1}((a,\infty))$

$\{x : f(x) > a\}$

$$\underbrace{g^{-1}((a,\infty)) = \underbrace{\left(f^{-1}((a,\infty))\right.}_{\text{meas}} \setminus \left.\underbrace{\left(\overbrace{\{f > a\}} \cap \{g \le a\}\right)}_{\to\ \text{null}}\right) \cup \underbrace{\left(\{g > a\} \cap \{f \le a\}\right)}_{\to\ \text{null}}}_{\text{measurable}}$$

If $f : D \to \mathbb{R}$ is measurable and $E \subseteq D$ is meas.

then $f|_E$ is measurable

(Exercise)

Thm: The measurable functions from a measurable set D to $\mathbb{R}$ form an algebra.

I.e. if $f, g : D \to \mathbb{R}$ are measurable then

1) cf is meas for all $c \in \mathbb{R}$

2) $f+g$ is meas

3) fg is meas

Pf: 1) Exercise or use 3)

2) Observe that

$$f(x)+g(x) > \alpha \quad \text{iff} \quad f(x) > \alpha - g(x)$$

$$\text{iff}$$

$$\exists\, r \in \mathbb{Q} \quad f(x) > r > \alpha - g(x)$$

$$\{f+g > \alpha\} = \bigcup_{r\in\mathbb{Q}} \{f > r\} \cap \{r > \alpha - g\}$$

$$= \underbrace{\bigcup_{r\in\mathbb{Q}} \underbrace{\underbrace{\{f>r\}}_{\text{meas.}} \cap \underbrace{\{g > \alpha - r\}}_{\text{meas.}}}_{\text{meas.}}}_{\text{meas.}}$$

countable union, so measurable.

3) First step: If f is measurable, f^2 is measurable.

$\alpha \geq 0$

$$\{f^2 > \alpha\} = \underbrace{\{f > \sqrt{\alpha}\}}_{\text{meas}} \cup \underbrace{\{f < -\sqrt{\alpha}\}}_{\text{meas}}$$

$\alpha < 0 \qquad \{f^2 > \alpha\} = D$

given f, g meas:

$$(f+g)^2 = f^2 + 2fg + g^2$$

$$fg = \frac{f^2 + g^2 - (f+g)^2}{2}$$

linear combo of meas functions.

Given f, g, measurable,

$h = \max(f, g)$ is measurable.

$$\{h > \alpha\} = \underbrace{\underbrace{\{f > \alpha\}}_{\text{meas}} \cup \underbrace{\{g > \alpha\}}_{\text{meas}}}_{\text{meas.}}$$

Given measurable $f_1, \ldots, f_n$

$\max(f_1, \ldots, f_n)$ is measurable.

$h(x) = \max(f(x), g(x))$

$\overline{\mathbb{R}}$ extended real numbers

$$\mathbb{R} \cup \{\infty, -\infty\}$$

$$D \subseteq \overline{\mathbb{R}}$$

$f : D \to \overline{\mathbb{R}}$ is measurable if

$$\underbrace{f^{-1}((a, \infty])}_{\{f > a\}} \text{ is measurable for all } a \in \overline{\mathbb{R}}$$

$b + \infty = \infty \quad \forall\, b \in \mathbb{R}$

$\infty + \infty = \infty$

$-\infty + \infty \Leftarrow$ uh oh

$\infty \cdot \infty = \infty$

$-\infty \cdot \infty = -\infty$

$0 \cdot \infty = 0$

$\cdot\, \infty$

$$\int_{\mathbb{R}} 0 = 0 \qquad 0 \cdot \infty = 0$$

$\{f_n\}_{n=1}^{\infty}$ $\quad f_n : D \to \overline{\mathbb{R}}$, measurable.

Claim: $\underbrace{\sup f_n}_{f}$ is measurable.

$$\{f > \alpha\} = \underbrace{\bigcup_n \{f_n > \alpha\}}$$

x $\quad f(x) \geq f_n(x) > \alpha$ for some n

$$f_n(x) \leq \alpha \quad \forall\, n,$$

α is an upper bound for $\{f_n(x);\ n=1,\ldots\infty\}$

$$f(x) = \sup_n f(x) \leq \alpha.$$

Exercise: $\inf\{f_n\}$ is measurable.

(use minus signs and the above)

$\limsup_n f_n = \inf_n \sup_{m \geq n} f_m$ is measurable.

$\{a_n\}$ $\limsup_{n \to \infty} a_n = \inf_n \sup_{m \geq n} a_m$

Same for $\liminf_n f_n$.

If $\{f_n\}$ $f_n : D \to \mathbb{R}$ measurable

and if $f_n \to f$ pointwise, then f is measurable.

We say $f_n \to f$ pointwise a.e. if there is a null set N and $f_n \to f$ pointwise on $D \setminus N$.

If $f_n \to f$ pointwise a.e. then f is measurable.

$$\tilde{f}_n = \begin{cases} f_n & \text{on } D \setminus N \\ 0 & \text{on } N \end{cases}$$

$$\underset{\text{meas.}}{\tilde{f}_n \xrightarrow[\text{p.w.}]{} \tilde{f}} = \begin{cases} f & \text{on } D \setminus N \\ 0 & \text{on } N \end{cases}$$

$f = \tilde{f}$ except on a subset of N.

$f = \tilde{f}$ a.e.

If $f : D \to \mathbb{R}$ is measurable

$$f^+ = \max(f, 0) \quad \text{is meas}$$

$$f^{-} = \min(f, 0) \quad \text{is meas}$$

$$|f| = f^+ - f^- \quad \text{is meas}$$

If $|f|$ is meas is f meas? (Exercise!)