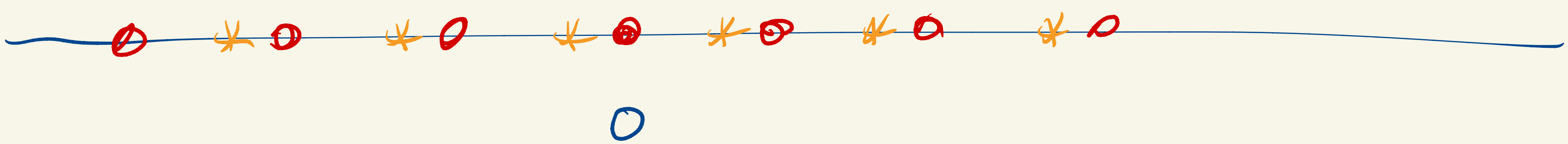
$(\mathbb{R}, +)$ is a group.

$\mathbb{Q} \subseteq \mathbb{R}$ is a subgroup.

Cosets:

$\mathbb{Z} \subseteq \mathbb{R}$ is a subgroup.

$$\underbrace{\mathbb{R}/\mathbb{Z}}_{\text{elements are cosets}}$$

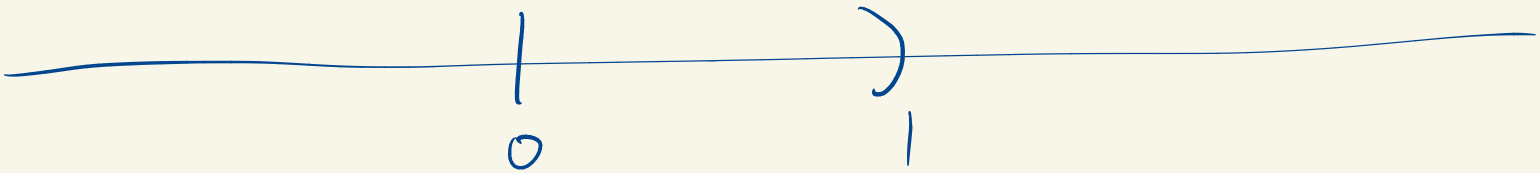


$\mathbb{Z} + t$

$t \in \mathbb{R}$

$$(\mathbb{Z} + a) + (\mathbb{Z} + b)$$

$$= \mathbb{Z} + (a + b)$$



$$x \dot{+} y = \begin{cases} x+y & x+y<1 \\ x+y-1 & x+y \geq 1 \end{cases} \qquad x, y \in [0,1)$$

$$x \in [0,1)$$

$$x^{-1} = 1 - x$$

Exercise: $\mathbb{Q} \cap [0,1)$ is a subgroup.

It has cosets.

Let $A \subseteq [0,1)$ be a set consisting of one representative from each coset. (Axiom of Choice!)

(A is uncountable)

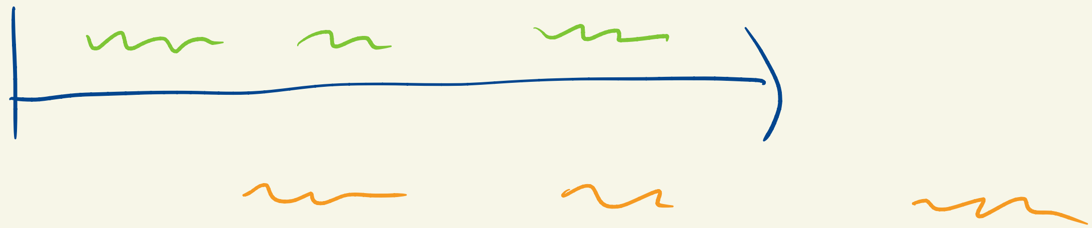
Claims: 1) Suppose $r_1, r_2 \in [0,1) \cap \mathbb{Q}$

If $A \dotplus r_1 \cap A \dotplus r_2 \neq \emptyset$ then $r_1 = r_2$

2) For all $x \in [0,1)$ there exists $r \in \mathbb{Q} \cap [0,1)$ such that $x \in A \dotplus r$

$$A \dotplus r = \{ a \dotplus r : a \in A \}$$

$$= \big(A \cap [0, 1-r)\big) + r + \big(A \cap [1-r, 1)\big) + r - 1$$



Suppose the claims hold.

Let $\rho : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ be a map that is translation invariant and countably additive.

Then either $\rho(A) = 0$ and $\rho([0,1)) = 0$ or $\rho(A) \neq 0$ and $\rho([0,1)) = \infty$.

Key: $\rho(A \dot{+} r) = \rho(A)$

$\Rightarrow$ finite additivity

$\Rightarrow$ monotonicity

$$
\begin{aligned}
p(A \dot{+} r) &= p\big((A \cap [0,1-r) + r) \cup (A \cap [1-r,1) + r - 1)\big) \\
&= p(A \cap [0,1-r) + r) + p(A \cap [1-r,1) + r - 1) \\
&= p(A \cap [0,1-r)) + p(A \cap [1-r,1)) \\
&= p\big((A \cap [0,1-r)) \cup (A \cap [1-r,1))\big) \\
&= p(A)
\end{aligned}
$$

$$
p([0,1)) = p\Big(\bigcup_{q \in \mathbb{Q} \cap [0,1)} A \dot{+} q \Big)
$$

countable additivity w/ Claim 1

Claim 2

$$= \sum_{q \in \mathbb{Q} \cap [0,1)} p(A \dotplus q)$$

$$= \sum_{q \in \mathbb{Q} \cap [0,1)} p(A)$$

If $p(A) = 0$ $\quad p([0,1)) = 0.$

If $p(A) \neq 0$ $\quad p([0,1)) = \infty.$

In fact, A is not measurable.

If it were the argument above would imply that

either $\quad m([0,1)) = 0 \quad$ or $\quad m([0,1)) = \infty.$

Proof of claims

1) Suppose $p \in A \overset{\circ}{+} r_1$ and $p \in A \overset{\circ}{+} r_2$.

Then there exists $a_1 \in A$ with $p = a_1 \overset{\circ}{+} r_1$

$a_2 \in A$ with $p = a_2 \overset{\circ}{+} r_2$

So $$a_1 \overset{\circ}{+} r_1 = a_2 \overset{\circ}{+} r_2$$

and $$a_1 = a_2 \overset{\circ}{+} \big(\underbrace{r_2 \overset{\circ}{+} (1 - r_1)}_{\in \mathbb{Q} \cap [0,1)} \big)$$

So a_1 and a_2 are in the same coset.

Hence $a_1 = a_2$ and $r_1 = r_2$.

2) Let $x \in [0,1)$.

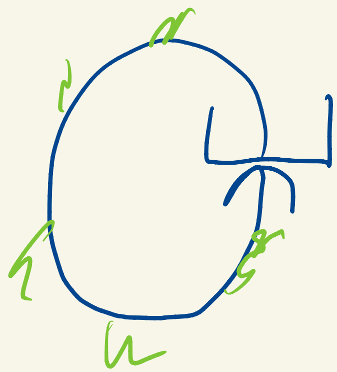
Its coset is $\mathbb{Q} \dot{+} x$.

There is an element $a \in A$ in this coset.

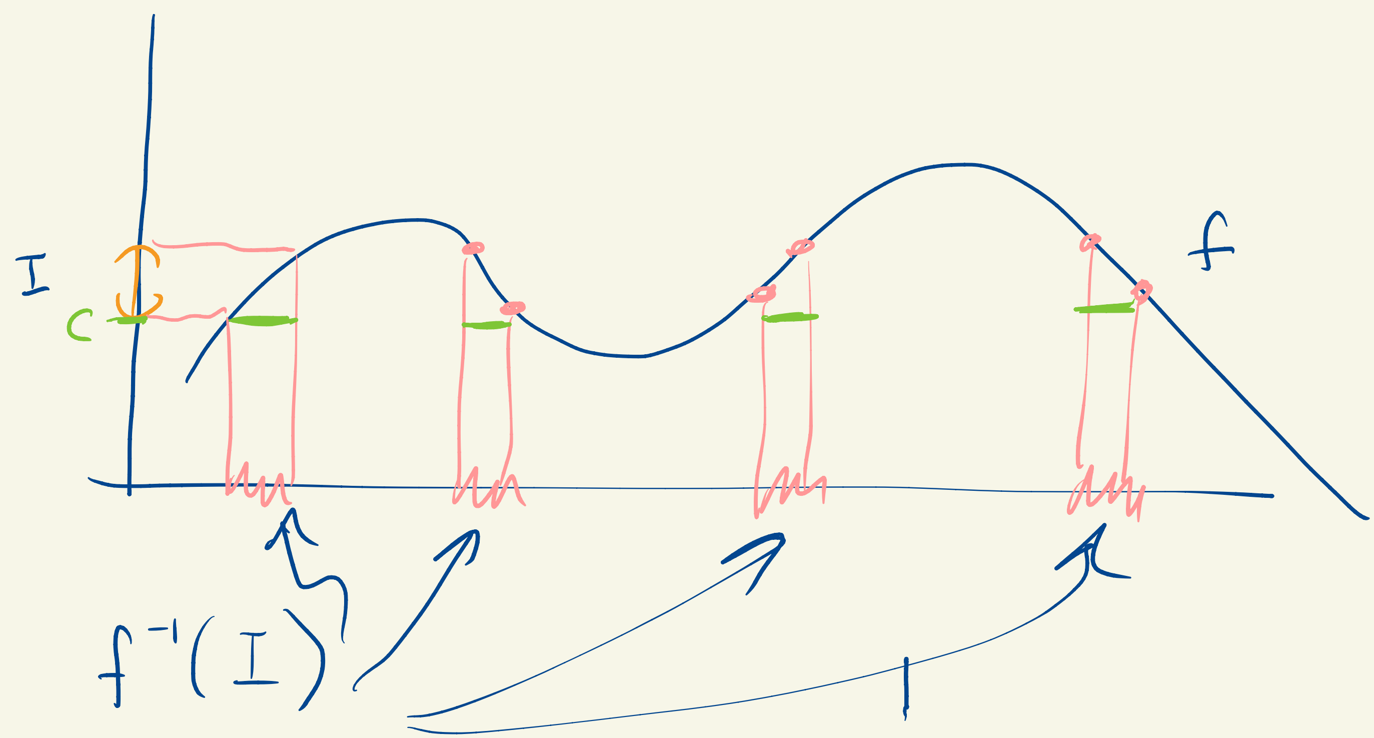
$$a = x \dot{+} q \quad \text{for some } q \in [0,1) \cap \mathbb{Q}.$$

$$x = a \dot{+} (1-q)$$

$$x \in A \dot{+} (1-q).$$



Measurable functions.



$c \cdot m(f^{-1}(I))$ approximates a part of $\int f$

We're going to want $f^{-1}(I)$ to be measurable.

Def: Let $D \subseteq \mathbb{R}$. We say that $f: D \to \mathbb{R}$ is (Lebesgue) measurable if

$$f^{-1}((a,\infty)) \in \mathcal{M} \text{ for all } a \in \mathbb{R}.$$

$$D = \bigcup_{n \in \mathbb{Z}_{\geq 0}} f^{-1}((-n,\infty)) \Rightarrow D \text{ is measurable.}$$

Exercise: Let $f: X \to Y$ (X, Y sets).

Suppose $\mathcal{A}$ is a σ-algebra of subsets of X.

Let $\mathcal{C} = \{ C \subseteq Y : f^{-1}(C) \in \mathcal{A} \}$.

Then $\mathcal{C}$ is a σ-algebra.

$$f^{-1}(C^c) = (f^{-1}(C))^c$$

$$f^{-1}\left(\bigcup_{n=1}^{\infty} C_n\right) = \bigcup_{n=1}^{\infty} f^{-1}(C_n)$$

Suppose f is measurable.

$$\mathcal{C} = \{ C \subseteq \mathbb{R} : f^{-1}(C) \in \mathcal{M} \}$$

Then $\mathcal{C}$ is a σ-algebra.

It contains each (a, ∞).

So it contains each $\bigcap_n (a - \frac{1}{n}, \infty) = [a, \infty)$

and $(-\infty, a]$ and $(-\infty, a)$.

So it also contains (a, b).

So it also contains all open sets.

$\mathcal{C}$ is a σ-algebra that contains all open sets.

$\mathcal{B}$, the borel sets, is the smallest σ-algebra that contains the open sets. So $\mathcal{B} \subseteq \mathcal{C}$.

That is, if f is measurable then

$$f^{-1}(B) \in \mathcal{M} \quad \text{for all Borel sets } B.$$

$f : \mathbb{R} \to \mathbb{R}$

If f is continuous, f is measurable

$$f^{-1}(\text{open}) = \text{open}$$

If f is monotone increasing, f is measurable

$$f^{-1}((a,\infty)) \text{ is an interval.}$$

If f is ~~lower~~ upper semicontinuous, f is measurable