Thm: TFAE $(E \subseteq \mathbb{R})$

1) E is measurable

2) $\forall\, \varepsilon > 0$ there exists an open set $U \supseteq E$ such that $m^*(U \setminus E) < \varepsilon$.

3) There exists a G_δ set $G \supseteq E$ such that

$$m^*(G \setminus E) = 0.$$

"Every measurable set is almost an open set."

Pf: We just proved $3) \Rightarrow 1)$.

$2) \Rightarrow 3)$

For each $n \in \mathbb{N}$ find an open set $O_n \supseteq E$ such that $m^*(O_n \setminus E) < \frac{1}{n}$.

Let $G = \bigcap_n O_n$, so G is a G_δ set.

Moreover $G \setminus E \subseteq O_n \setminus E$ for all n.

So, by monotonicity, $m^*(G \setminus E) \leq m^*(O_n \setminus E) < \frac{1}{n}$ for all n.

So $m^*(G \setminus E) = 0$.

1) $\Rightarrow$ 2)

First, suppose $m^*(E) < \infty$. Let $\varepsilon > 0$. Let $\{I_n\}$ be a measuring cover of E such that $\sum_n \ell(I_n) < m^*(E) + \varepsilon$.

Let $U = \bigcup_n I_n$ so $U \supseteq E$.

Because E is measurable

$$m^*(U) = m^*(U \cap E) + m^*(U \cap E^c)$$
$$= m^*(E) + m^*(U \setminus E).$$

On the other hand, by countable subadditivity,

$$m^*(U) \leq \sum_n \ell(I_n) < m^*(E) + \varepsilon.$$

Hence

$$m^*(E) + m^*(U \setminus E) < m^*(E) + \varepsilon.$$

Since $m^*(E) < \infty$, $m^*(U \setminus E) < \varepsilon$.

Now let $E \subseteq \mathbb{R}$ be measurable and otherwise arbitrary.

For each n let $E_n = [-n, n] \cap E$. So each E_n is measurable and has finite measure. Find open sets $O_n \supseteq E_n$ such that $m^*(O_n \setminus E_n) < \varepsilon/2^n$.

Let $O = \bigcup_n O_n$, so O is open and $O \supseteq E$.

Now $m^*(O \setminus E) = m^*\left(\left(\bigcup_n O_n\right) \setminus E\right)$

$$\leq \sum_n m^*(O_n \setminus E) \qquad O_n \setminus E \subseteq O_n \setminus E_n$$

$$\leq \sum_n m^*(O_n \setminus E_n)$$

$$< \sum_n \varepsilon/2^n = \varepsilon.$$

Exercise: TFAE

1) E is measurable,

2) $\forall \varepsilon > 0$ there is a closed set $F \subseteq E$ such that

$$m^*(E \setminus F) < \varepsilon.$$

3) there is an F_σ set F with $F \subseteq E$ and

$$m^*(E \setminus F) = 0.$$

Exercise: E is measurable iff for all $\varepsilon > 0$ there exists an open set U and a closed set F such that $U \supseteq E \supseteq F$ and $m^*(U \setminus F) < \varepsilon$.

Exercise: Suppose $m^*(E) < \infty$. Then E is measurable iff for all $\varepsilon > 0$ there exists a finite collection of open intervals $\{I_k\}_{k=1}^n$ such that $m^*(E \,\Delta\, U) < \varepsilon$ where $U = \bigcup_k I_k$.

set diff.

$$(E \setminus U) \cup (U \setminus E)$$

Lebesgue measure possesses a kind of continuity.

$$E_n \to E \implies m(E_n) \to m(E)$$

?

$$E_1 \subseteq E_2 \subseteq E_3 \subseteq \cdots \qquad E = \bigcup_k E_k.$$

Claim: $m(E_k) \to m(E)$

"no extra length can appear in the limit"

Continuity from below

$$F_k = E_k \setminus E_{k-1}.$$

Let $F_1 = E_1$

Let $F_2 = E_2 \setminus E_1$

Let $F_3 = E_3 \setminus (E_1 \cup E_2) = E_3 \setminus E_2$

The F_k's are disjoint and $E_k = \bigcup_{j=1}^{k} F_k$.

So $m(E_k) = \sum_{j=1}^{k} m(F_j) \longrightarrow \sum_{j=1}^{\infty} m(F_j) = m(E)$

finite additivity

countable additivity

$\bigcup F_j = E$

could be ∞.

Does continuity from above hold?

E_n, measurable, $E_{n+1} \subseteq E_n$. $E = \bigcap E_n$

Does $m(E_n) \to m(E)$

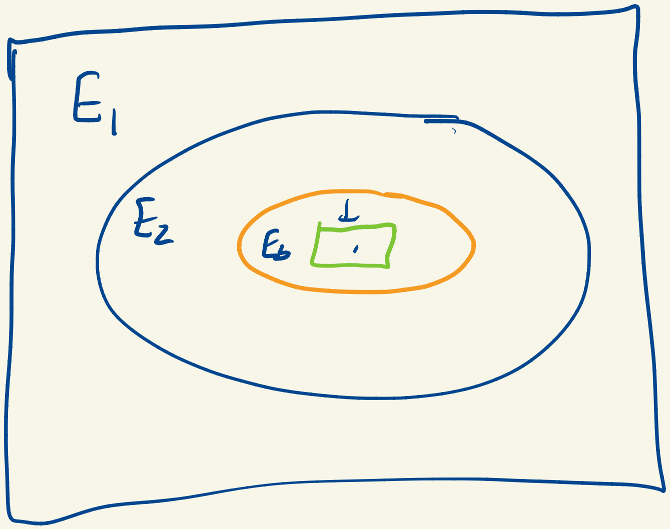
$E_n = (n, \infty)$ $\qquad$ $m(E_n) = \infty$

$$E = \bigcap E_n = \emptyset \qquad m(\emptyset) = 0$$

Continuity from above holds if we rule out this phenomenon.

Prop: Let $\{E_k\}_{k=1}^{\infty}$ be a collection of measurable sets with $E_{k+1} \subseteq E_k$ for all k and such that $m(E_1) < \infty$.

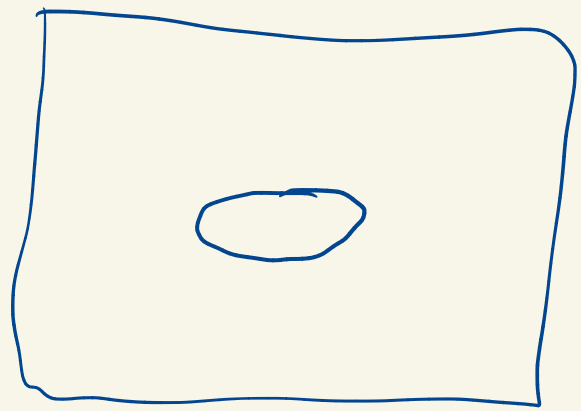
Then $\lim_{k \to \infty} m(E_k) = m(E)$ where $E = \bigcap E_k$.



$F_k = E_1 \setminus E_k$

$\emptyset$,

,



F_1 F_2 F_3 $\dashrightarrow$

The F_k's are increasing so

$$\underline{m(F_k)} \to m\left(\bigcup_k F_k\right)$$

$$m(F_k) = m(E_1 \setminus E_k)$$

$$m(E_1) = m(E_1 \setminus E_k) + m(E_k)$$

$m(E_k) < \infty$

$$m(E_1) - m(E_k) = m(E_1 \setminus E_k)$$

$$m(E_1) - m(E_k) \to m\left(\bigcup_k F_k\right)$$

$$\bigcup_k F_k = \bigcup_k (E_1 \setminus E_k) = \bigcup_k (E_1 \cap E_k^c)$$

$$= E_1 \cap \left(\bigcup_k E_k^c\right)$$

$$= E_1 \cap \left(\bigcap_k E_k\right)^c$$

$$= E_1 \setminus E$$

$$m\left(\bigcup_k F_k\right) = m(E_1) - m(E)$$

$$m(E_1) - m(E_k) \longrightarrow m(E_1) - m(E)$$

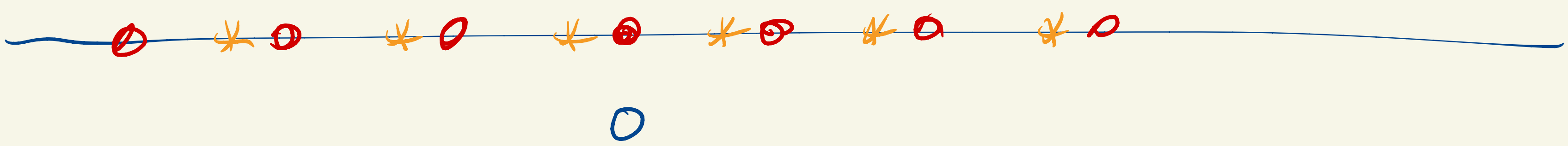
$$m(E_k) \longrightarrow m(E) \qquad (m(E_1) < \infty).$$

$(\mathbb{R}, +)$ is a group.

$\mathbb{Q} \subseteq \mathbb{R}$ is a subgroup.

Cosets:

$\mathbb{Z} \subseteq \mathbb{R}$ is a subgroup.



$\mathbb{Z} + t$

$t \in \mathbb{R}$