Measurable sets and set operations.

Def: A collection $\mathcal{A}$ of subsets of X is an <u>algebra</u> of sets if whenever $A, B \in \mathcal{A}$

1) $A \cup B \in \mathcal{A}$
2) $A \cap B \in \mathcal{A}$
3) $A^c \in \mathcal{A}$

suffices

$A \cap B \qquad (A^c \cup B^c)^c$

Is $X \in \mathcal{A}$? If $\mathcal{A} \neq \emptyset$ $\quad A \in \mathcal{A}$, $A^c \in \mathcal{A}$ $\quad \underbrace{A \cup A^c}_{X} \in \mathcal{A}$

More generally:

Def: A collection $\mathcal{A}$ of subsets of X is a <u>σ-algebra</u> of sets if

1) If $\{A_k\}_{k=1}^{\infty}$ is a collection in $\mathcal{A}$, $\bigcup_{k=1}^{\infty} A_k \in \mathcal{A}$

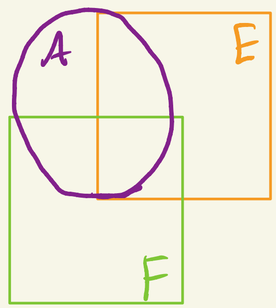
2) If $A \in \mathcal{A}$ then $A^c \in \mathcal{A}$.

(and hence if $\{A_k\}_{k=1}^{\infty}$ is in $\mathcal{A}$, $\bigcap_k A_k \in \mathcal{A}$)

We aim to show that $\mathcal{M}$ is a σ-algebra.

(and moreover it contains the open sets)

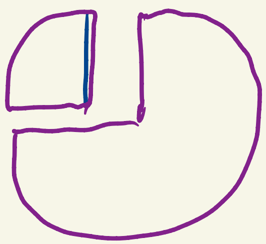
Step 1: Show $\mathcal{M}$ is an algebra.



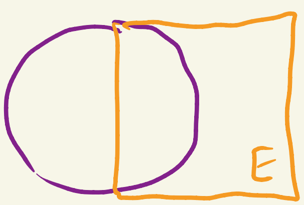
$E, F \in m$

Want $E \cup F$ is measurable.

$E \cup F$ carves well

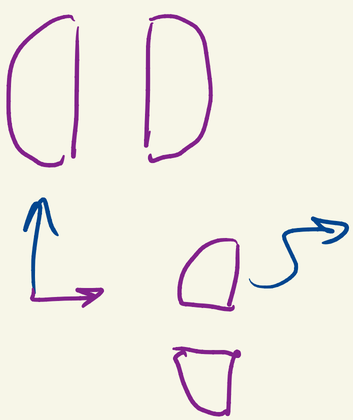


$$m^*(A \cap (E \cup F)) + m^*(A \cap (E \cup F)^c) = m^*(A)$$



$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$$

$A \cap E^c$



$$m^*(A \cap E^c) = m^*(A \cap E^c \cap F) + m^*(A \cap E^c \cap F^c)$$

$A \cap E^c \cap F^c$

$A \cap E^c \cap F$

$$m^*(A\cap(E\cup F)) =$$

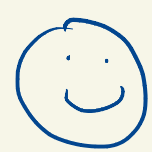
$$m^*(A\cap(E\cup F)\cap E) + m^*(A\cap(E\cup F)\cap E^c)$$

$$= m^*(A\cap E) + m^*(A\cap F\cap E^c)$$

$$m^*(A) = m^*(A\cap E) + m^*(A\cap E^c)$$

$$= m^*(A\cap E) + m^*(A\cap E^c\cap F) + m^*(A\cap E^c\cap F^c)$$

$$= m^*(A\cap(E\cup F)) + m^*(A\cap(E\cup F)^c)$$



$\mathcal{M}$ is an algebra!

Lemma: Suppose $\{E_i\}_{i=1}^n$ are disjoint and measurable. Then for all $A \subseteq \mathbb{R}$,

$$m^*\left(A \cap \bigcup_{i=1}^n E_i\right) = \sum_{i=1}^n m^*(A \cap E_i)$$

Pf: The proof is by induction. The case $n=1$ is obvious.

Suppose the result holds for some n.

Consider a disjoint collection $\{E_i\}_{i=1}^{n+1}$ of measurable sets.

Let $A \subseteq \mathbb{R}$. Then

$$m^*\left(A \cap \bigcup_{i=1}^{n+1} E_i\right) = m^*\left(A \cap \bigcup_{i=1}^{n+1} E_i \cap E_{n+1}\right)$$

$$+ m^*\left(A \cap \bigcup_{i=1}^{n+1} E_i \cap E_{n+1}^c\right)$$

$$= m^*\left(A \cap E_{n+1}\right) + m^*\left(A \cap \bigcup_{i=1}^{n} E_i\right)$$

$$= m^*(A \cap E_{n+1}) + \sum_{i=1}^{n} m^*(A \cap E_i)$$

$$= \sum_{i=1}^{n+1} m^*(A \cap E_i) \qquad \checkmark$$

Prop: If $\{E_i\}_{i=1}^{\infty}$ are disjoint and measurable then

$$\bigcup_{i=1}^{\infty} E_i \in \mathcal{M}.$$

Pf: Let $A \subseteq \mathbb{R}$. Let $E = \bigcup_{i=1}^{\infty} E_i$. It suffices to show

$$m^*(A \cap E) + m^*(A \cap E^c) \leq m^*(A).$$

For each n,

$$m^*(A) = m^*\left(A \cap \left(\bigcup_{i=1}^{n} E_i\right)\right) + m^*\left(A \cap \left(\bigcup_{i=1}^{n} E_i\right)^c\right)$$

(measurable sets are an algebra)

$$\geq m^*\left(A \cap \left(\bigcup_{i=1}^{n} E_i\right)\right) + m^*(A \cap E^c)$$

(monotonicity)

$$= \sum_{i=1}^{n} m^*(A \cap E_i) + m^*(A \cap E^c).$$

(lemma)

This holds for all n, so

$$m^*(A) \geq \sum_{i=1}^{\infty} m^*(A \cap E_i) + m^*(A \cap E^c)$$

$$\geq m^*(A \cap E) + m^*(A \cap E^c)$$

(countable subadditivity)

What if we have a collection $\{F_i\}_{i=1}^{\infty}$

of measurable, not necessarily disjoint sets?

$$E_n = F_n \setminus \bigcup_{k=1}^{n-1} F_k \text{, which is measurable}$$

E_n's are disjoint

$$\underbrace{\bigcup E_n}_{} = \bigcup F_n$$

$\uparrow \qquad\qquad \uparrow$

$\in \mathcal{M} \Rightarrow \in \mathcal{M}$

$\mathcal{M}$ is a σ-algebra.

Exercise:

If $E \in \mathcal{M}$ then

$E + t \in \mathcal{M} \ \forall t \in \mathbb{R}$.

and $rE \in \mathcal{M}$

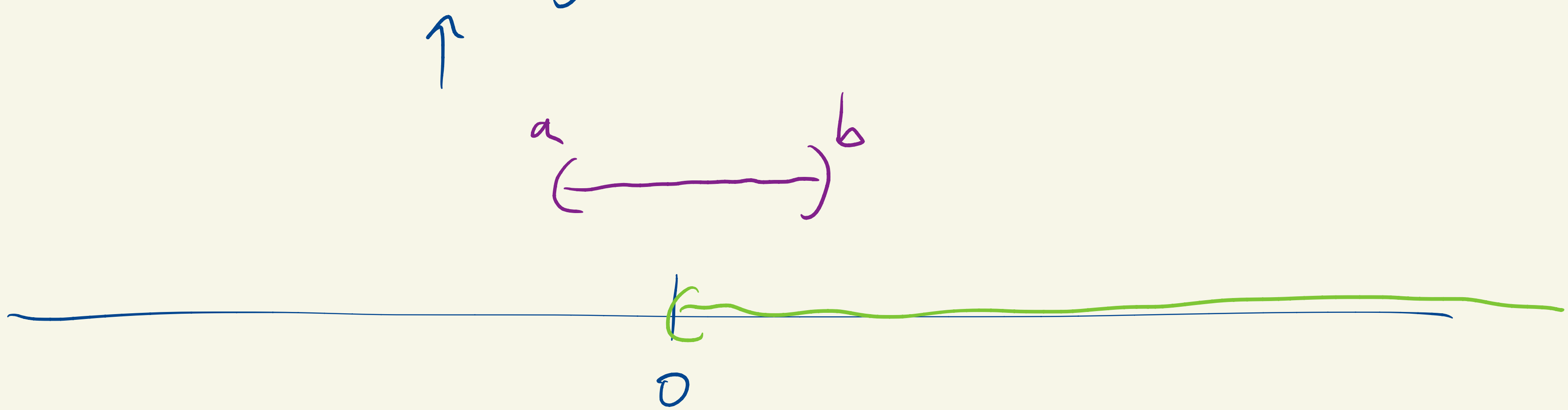
for all $r \in \mathbb{R}$.

$\mathcal{M}$ is a σ-algebra and

m satisfies 1) — 6) (and hence also 7))

Measurable sets and topology.

$(0, \infty)$



$(a, 0]$, $(0, b)$ $\qquad m^*((a,0]) + m^*((0,b])$

$= 0 - a + b - 0 = b - a = m^*((a,b))$

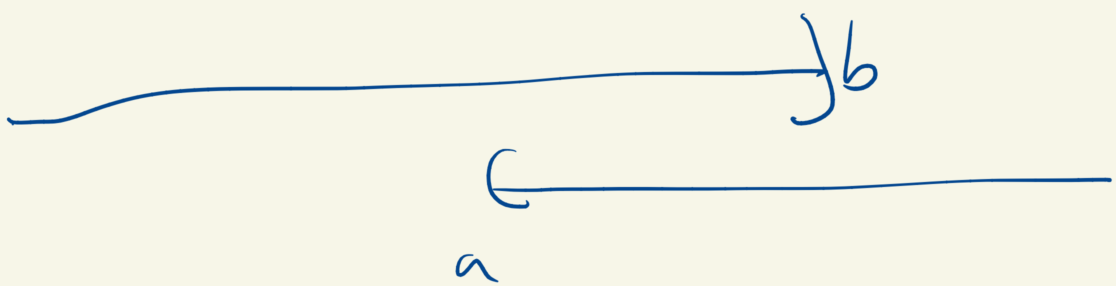
$(0,\infty)$ is measurable.

(a,∞) is measurable $\forall a\in\mathbb{R}$.

$(-\infty, b]$ is measurable $\forall b\in\mathbb{R}$

$(-\infty, b)$ is measurable $\forall b\in\mathbb{R}$

(a,b) are measurable $\forall a<b$.



All intervals are measurable.

Every open set is a countable union of
open intervals.

Every open set is measurable.

Every closed set is measurable

Every countable intersection of open sets is measurable.

G_δ sets gebeit

↳ durchschnitt

Every countable union of closed sets is measurable

F_σ sets

↳ summe

Exercise: Let X be a set. Let $\{\mathcal{A}_\alpha\}_{\alpha \in I}$ be a collection of σ-algebras in X.

Then $\bigcap_{\alpha \in I} \mathcal{A}_\alpha$ is again a σ-algebra in X.

Exercise: Let $\mathcal{C}$ be a collection of subsets of X. There is a unique smallest σ-algebra containing $\mathcal{C}$. (It is called the σ-algebra generated by $\mathcal{C}$).

The σ-algebra generated by the open sets

M $\mathbb{R}$ is known as $\mathcal{B}$, the Borel sets.

(b.t.w. this is strict)

$$\mathcal{B} \subseteq \underline{\mathcal{M}}.$$

$\hookrightarrow$ open sets,
closed sets

G_δ, F_σ, $G_{\delta\sigma}$, $F_{\sigma\delta}$