What remains to show

5) finite additivity

6) countable subadditivity

7) countable additivity

$A = \bigcup A_k$

A_k's disjoint

$\ell(A) = \sum \ell(A_k)$

m^*

We claimed you can't have 1), 2), 5),

(1), 2), 7)

Prop: m^* is countably subadditive.

Pf: Let $\{A_k\}_{k=1}^{\infty}$ be a sequence of subsets of $\mathbb{R}$.

Let $\varepsilon > 0$.

For each k pick a measuring cover $\{I_{j,k}\}_{j=1}^{\infty}$

of open intervals such that $\sum_{j=1}^{\infty} \ell(I_{j,k}) \leq m^*(A_k) + \frac{\varepsilon}{2^k}$.

Observe that $\{I_{j,k}\}_{j,k=1}^{\infty}$ is a measure cover of

$\bigcup A_k$. Moreover $\sum_{j,k} \ell(I_{j,k}) = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \ell(I_{j,k})$

$$\leq \sum_{k=1}^{\infty} \left(m^*(A_k) + \frac{\varepsilon}{2^k} \right)$$

$$= \left[\sum_{k=1}^{\infty} m^*(A_k) \right] + \varepsilon$$

So $m^*(A) \leq \sum_{k=1}^{\infty} m^*(A_k) + \varepsilon$ for any $\varepsilon > 0$.

$\square$

Hence

$$m^*(A) \leq \sum_{k=1}^{\infty} m^*(A_k).$$

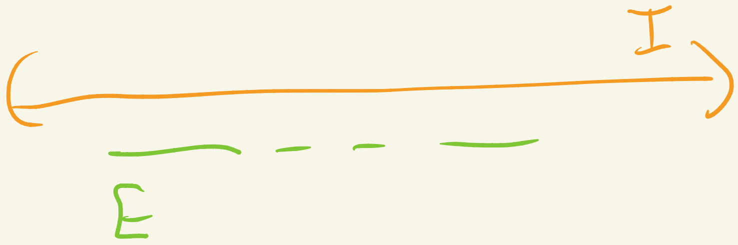
If m^* is not finitely additive it must mean there are disjoint sets A, B with

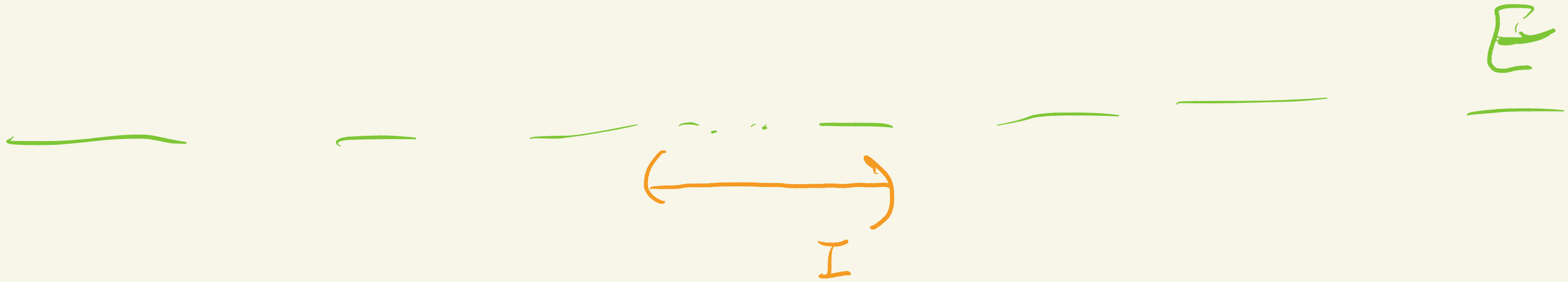
$$m^*(A \cup B) < m^*(A) + m^*(B)$$

How can you tell if m^* is assigns "too much" length to some set E,

If E lives inside an interval I we could look at E and $I \setminus E$

$$m^*(E) + m^*(I \setminus E) > \ell(I)$$





$$m^*(E \cap I) + m^*(I \cap E^c) > \ell(I)$$

Def: A set $E \subseteq \mathbb{R}$ satisfies condition CC′ if for all intervals I

$$m^*(E \cap I) + m^*(E^c \cap I) = \ell(I).$$

Def: A set $E \subseteq \mathbb{R}$ satisfies condition CC if for all sets A

$$m^*(E \cap A) + m^*(E^c \cap A) = m^*(A).$$

[We showed $\ell(I) = m^*(I)$ if I is closed and bounded.
Use this and monotonicity to show that the same formula holds for any interval]

Clearly CC $\Rightarrow$ CC'

Prop: A set $E \subseteq \mathbb{R}$ satisfies CC iff it satisfies CC'

Pf: Suppose E satisfies CC'.

Let $A \subseteq \mathbb{R}$. Let $\varepsilon > 0$.

Let $\{I_k\}$ be a measuring cover of A

with $\sum_k \ell(I_k) \leq m^*(A) + \varepsilon$.

For each I_k, $m^*(E \cap I_k) + m^*(E^c \cap I_k) = \ell(I_k)$.

Observe that $A \cap E \subseteq \bigcup_k (I_k \cap E)$ and countable

subadditivity implies

$$m^*(A \cap E) \leq \sum_k m^*(I_k \cap E).$$

Similarly

$$m^*(A \cap E^c) \leq \sum_k m^*(I_k \cap E^c)$$

Hence $m^*(A \cap E) + m^*(A \cap E^c) \leq \sum_k \left(m^*(I_k \cap E) + m^*(I_k \cap E^c) \right)$

$$= \sum_k \ell(I_k)$$

$$\leq m^*(A) + \varepsilon.$$

This holds for all $\varepsilon > 0$, so

$$m^*(A \cap E) + m^*(A \cap E^c) \leq m^*(A).$$

But the reverse inequality holds by countable subadditivity.

$\square$

Def: A set $A \subseteq \mathbb{R}$ is <u>measurable</u> if it satisfies condition CC (or equivalently CC').

Let E_1 and E_2 be measurable sets.
disjoint.

$$m^*(E_1 \cup E_2) = m^*((E_1 \cup E_2) \cap E_2) + m^*((E_1 \cup E_2) \cap E_2^c)$$

$$= m^*(E_2) + m^*(E_1)$$

This looks like finite additivity. But is $E_1 \cup E_2$ measurable?

Are there any measurable sets?

$$m^*(\phi) = 0$$

$$m^*(A \cap \phi) + m^*(A \cap \phi^c) = m^*(\phi) + m^*(A) = m^*(A).$$

Notice that if E is measurable so is E^c.

$$m^*(A \cap E^c) + m^*(A \cap (E^c)^c) = m^*(A \cap E^c) + m^*(A \cap E)$$
$$= m^*(A).$$

$\mathbb{R}$ is measurable!

Def

A set $E \subseteq \mathbb{R}$ is <u>null</u> if $m^*(E) = 0$.

Every null set is measurable.

Let $A \subseteq \mathbb{R}$.

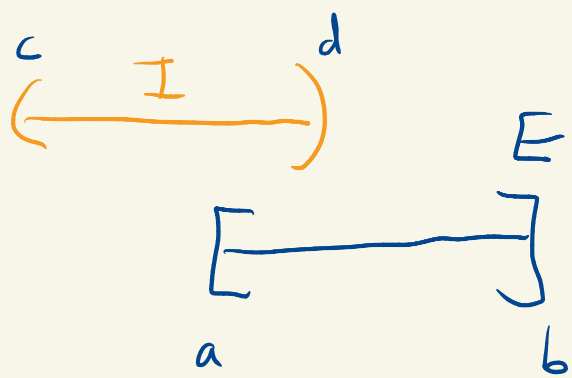
$$m^*(N \cap A) + m^*(N^c \cap A) \leq m^*(N) + m^*(A)$$
$$= m^*(A)$$

But $m^*(N\cap A)+m^*(N^c\cap A)\geq m^*(A)$ by countable subadditivity.

Hence $m^*(N\cap A)+m^*(N^c\cap A)=m^*(A)$.

Intervals are all measurable.

We'll show they satisfy condition CC'.



I

$m^*(E\cap I)+m^*(E^c\cap I)$

$E\cap I=[a,d)$

$E^c\cap I=(c,a)$

$$m^*(E\cap I)+m^*(E^c\cap I)=\ell([a,d))+\ell((c,a))$$
$$=d-a+a-c$$
$$=d-c=\ell(I)$$

The set of measurable subsets of $\mathbb{R}$ is denoted by $\mathcal{M}$.

We denote $m^*\Big|_{\mathcal{M}} = m$ and call it Lebesgue measure.

We want to show

1) $m([a,b]) = \ell([a,b])$

2) If $E \in \mathcal{M}$ and $t \in \mathbb{R}$ then $E+t \in \mathcal{M}$ and $m(E+t) = m(E)$

3) If $E \in \mathcal{M}$ and $r \in \mathbb{R}$ then $rE \in \mathcal{M}$ and $m(rE) = |r|\, m(E)$

4) If $E, F \in \mathcal{M}$ and $E \subseteq F$ then

$$m(E) \leq m(F)$$

5) If $E, F \in \mathcal{M}$ and are disjoint then

$E \cup F \in \mathcal{M}$ and $m(E \cup F) = m(E) + m(F)$

6) If $\{E_k\}$ is a sequence in $\mathcal{M}$ then

$\bigcup E_k \in \mathcal{M}$ and $m(\bigcup E_k) \leq \sum m(E_k)$

7) If $\{E_k\}$ is a sequence of disjoint <u>sets</u> in $\mathcal{M}$ then $m(\bigcup E_k) = \sum m(E_k)$