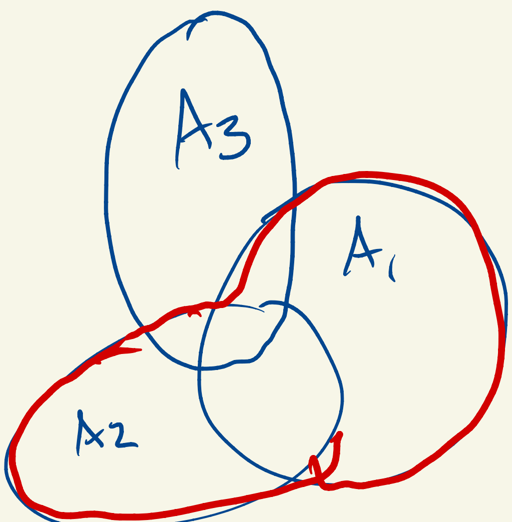
Warmup:

Lemma: Let $\{A_k\}_{k=1}^{\infty}$ be a collection of subsets of some set A. Then there is a collection $\{B_k\}_{k=1}^{\infty}$ of disjoint subsets of A with

- $B_k \subseteq A_k \quad \forall k$
- $\bigcup_{k=1}^{n} B_k = \bigcup_{k=1}^{n} A_k$ for all n.



$B_1 = A_1$

$B_1 \cup B_2 = A_1 \cup A_2$

$B_1 \cap B_2 = \phi$

$B_2 = A_2 \setminus A_1$

$B_3 = A_3 \setminus \left(\bigcup_{k=1}^{2} A_k\right)$

$$B_1 = A_1 \qquad B_n = A_n \setminus \bigcup_{k=1}^{n-1} A_k$$

Pf (of prop). Suppose f is finitely additive and countably subadditive. (Job: show f is countable additive)

Consider a disjoint collection $\{A_k\}_{k=1}^{\infty}$.

For each $n \in \mathbb{N}$

$$f\left(\bigcup_{k=1}^{\infty} A_k\right) \underset{\substack{\uparrow \\ \text{monotinicity} \\ \text{from finite additivity}}}{\geq} f\left(\bigcup_{k=1}^{n} A_k\right) \underset{\substack{\uparrow \\ \text{finite} \\ \text{additivity (disjoint!)}}}{=} \sum_{k=1}^{n} f(A_k)$$

Hence $f\left(\bigcup_{k=1}^{\infty} A_k\right) \geq \sum_{k=1}^{\infty} f(A_k)$.

But by countable subadditivity $\sum_{k=1}^{\infty} f(A_k) \geq f\left(\bigcup_{k=1}^{\infty} A_k\right)$.

Hence $\sum_{k=1}^{\infty} f(A_k) = f\left(\bigcup_{k=1}^{\infty} A_k\right)$.

Converse: HW

Minimal sets of desired properties

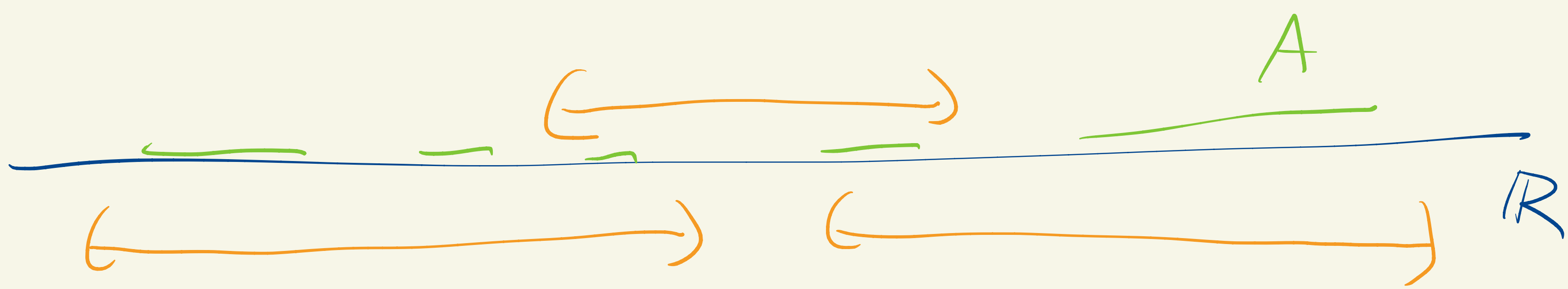
1)	1)	Bad news:
2)	2)	You can't have all of
3)	3)	1), 2), and 7).
7)	5)	
	6)	

Def: Let $A \subseteq \mathbb{R}$. A <u>measuring cover</u> of A is a countable collection $\{I_n\}_{n=1}^{\infty}$ of open intervals (possibly empty) such that

$$A \subseteq \bigcup_{n=1}^{\infty} I_n .$$

Def: Let $A \subseteq \mathbb{R}$. The <u>Lebesgue outer measure</u> $m^*(A)$ is

$$\inf\left\{ \sum_n \ell(I_n) : \{I_n\} \text{ is a measuring cover of } A \right\}.$$



To what extent is m^* our ideal length function?

monotonicity is pretty clear. $A \subseteq B$

every measuring cover of B

is also a measuring cover of A.

translation invariance (HW)

scaling covariance (Exercise)

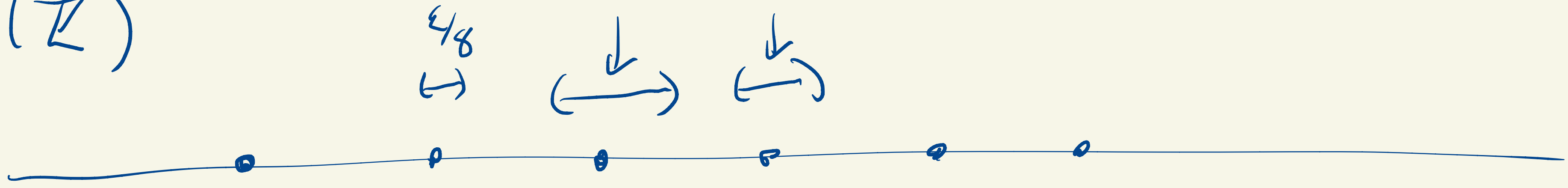
harder: $m^*([a,b]) = b-a$

$\{(a-\varepsilon, b+\varepsilon)\}$

$\hookrightarrow \ell((a-\varepsilon, b+\varepsilon)) = b-a+2\varepsilon$

$m^*([a,b]) \leq b-a$

$m^*(\mathbb{Z})$ $\qquad \varepsilon > 0$



$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1$$

$$m^*(\mathbb{Z}) \le \varepsilon \quad \forall\, \varepsilon > 0.$$

$$m^*(\mathbb{Z}) \ge 0$$

$$\Rightarrow \quad m^*(\mathbb{Z}) = 0$$

In fact if $A \subseteq \mathbb{R}$ is countable, then $m^*(A) = 0$.

Pf: Let $A = \{a_k\}_{k=1}^{\infty}$. Let $\varepsilon > 0$. For each k, let $I_k = \left(a_k - \frac{\varepsilon}{2^{k+1}},\ a_k + \frac{\varepsilon}{2^{k+1}}\right)$ so $\ell(I_k) = \frac{\varepsilon}{2^k}$.

Then $\{I_k\}$ is a measuring cover for A.

Hence $m^*(A) \le \sum_{k=1}^{\infty} \ell(I_k) = \sum_{k=1}^{\infty} \varepsilon/2^k = \varepsilon.$

This is true for all $\varepsilon > 0$ so $m^*(A) \le 0$.

Since $m^*(A) \ge 0$ we have $m^*(A) = 0$.

$m^*(\mathbb{Q}) = 0$

$m^*(\Delta) = 0$ (HW)

Prop: If $a < b$ then

$$m^*([a,b]) = b-a.$$

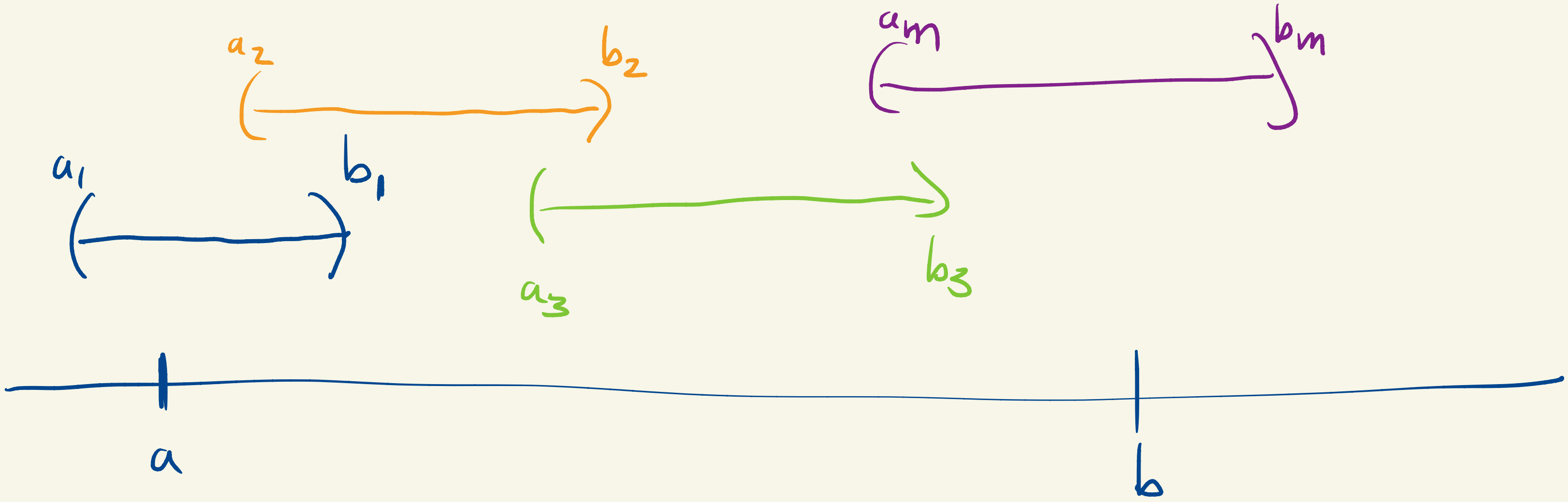
Pf: We have already shown $m^*([a,b]) \leq b-a$.

So it suffices to show the reverse inequality.

Let $\{I_k\}_{k=1}^{\infty}$ be a measuring cover of $[a,b]$.

Since the interval is compact we can extract a finite subcover $\{J_k\}_{k=1}^{n}$, which is also a measuring cover of $[a,b]$. Since $\sum_{k=1}^{n} \ell(J_k) \leq \sum_{k=1}^{\infty} \ell(I_k)$

it suffices to show $\sum_{k=1}^{n} \ell(J_k) \geq b-a$.



Without loss of generality, $a \in J_1 = (a_1, b_1)$.

If $b \in J_1$ clearly $\underbrace{\ell([a,b])}_{b-a} \leq \ell(J_1) \leq \sum_{k=1}^{n} \ell(J_k)$.

Otherwise, WLOG, $b_1 \in J_2 = (a_2, b_2)$.

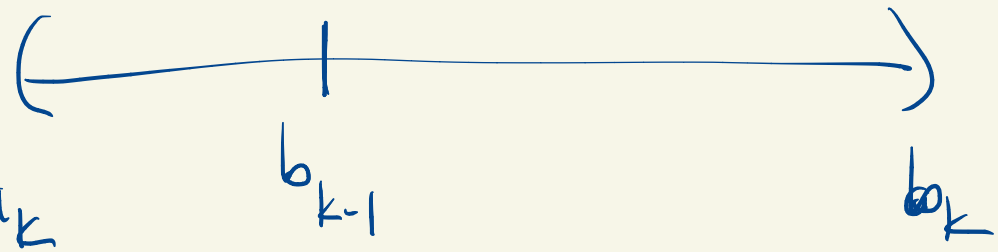
Continuing this procedure we can assume

that we have intervals $J_1, J_2, \ldots, J_m$

with $J_k = (a_k, b_k)$ and $b_k \in J_{k+1}$ for $k = 1, \dots, m-1$

and $b \in J_m$.

Observe that for each k,



$$b_k - a_k \geq b_k - b_{k-1}.$$

So

$$\sum_{k=1}^{m} \ell(J_k) = \sum_{k=1}^{m} b_k - a_k \geq (b_1 - a) + (b_2 - b_1) + (b_3 - b_2) + \cdots (b - b_{m-1})$$

$= b - a.$

Hence $\sum_{k=1}^{\infty} \ell(I_k) \geq b-a$ as well, so

$m^*([a,b]) \geq b-a$ as well.

Next class: m^* is countably subadditive.