Then by monotonicity $\int_a^b (0 \vee H - 0 \vee h) \leq \int_a^b (H-h) < \varepsilon.$

Prop: Suppose (f_n) is a sequence in $R[a,b]$ converges uniformly to some f. Then $f \in R[a,b]$ and

$$\int_a^b f_n \longrightarrow \int_a^b f.$$

Pf: We first show that $f \in R[a,b]$.

Let $\varepsilon > 0$. Pick some N such that $|f - f_n| < \varepsilon$ on $[a,b]$.

if $n \geq N$

Now pick some $n \geq N$. Find step functions $h \leq f_n \leq H$ such that $\int_a^b H - h < \varepsilon$. Observe that $h - \varepsilon$ and $H + \varepsilon$

are step functions and $h - \varepsilon \le f \le H + \varepsilon$.

$$h-\varepsilon \le f_n - \varepsilon < f < f_n + \varepsilon \le H+\varepsilon$$

Moreover $\int_a^b (H+\varepsilon) - (h-\varepsilon)$

$$= 2\varepsilon(b-a) + \int_a^b H - h$$

$$< 2\varepsilon(b-a) + \varepsilon$$

$$= (1 + 2(b-a))\varepsilon.$$

Hence f is Riemann integrable.

The proof that $\int_a^b f_n \to \int_a^b f$ is now identical to our earlier proof assuming f is continuous.

Remark: The FTC holds for continuous integrands (see your undergrad text)

Deficiencies of the Riemann integral.

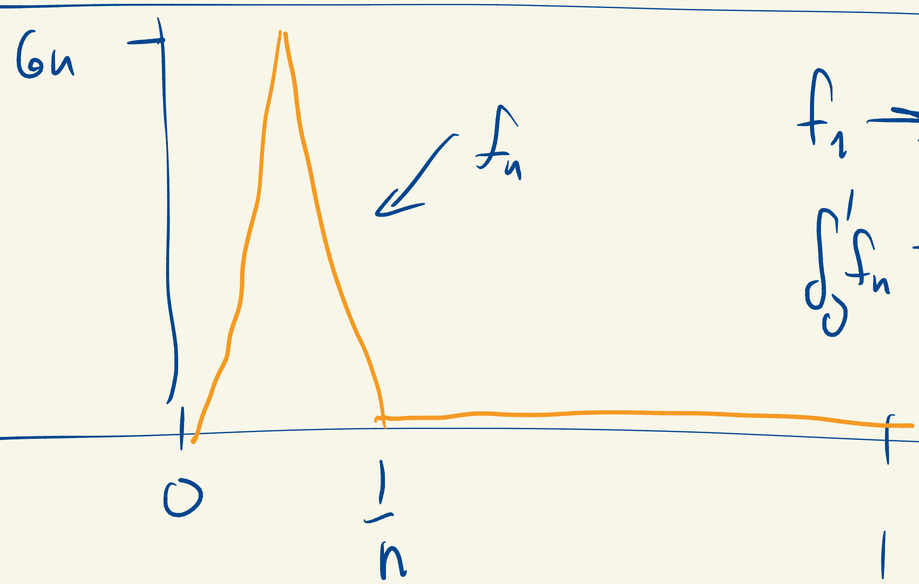
1) Unbounded functions. $\int_0^1 \frac{1}{\sqrt{x}}\,dx$

2) Unbounded domains $\int_1^\infty \frac{1}{x^2}\,dx$

3) Convergence issues

Uniform convergence is rare, but the set of Riemann integrable functions is not closed under pointwise convergence.

$\chi_{\mathbb{Q}}$ on $[0,1]$ is a pointwise limit of step functions.



$f_n \to 0$

$\int_0^1 f_n = 3 \quad \forall n$

$\int_0^1 f_n \not\to 0 = \int_0^1 0$

Arzela's Dominated Convergence Theorem

If (f_n) is a sequence in $R[a,b]$ and if there exists $M \in \mathbb{R}$ with $|f_n| \le M$ for all n and, if $f_n \to f$ pointwise and $\boxed{\text{if } f \in R[a,b]}$ then

$$\int_a^b f_n \to \int_a^b f.$$

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f will only take on the values $0 + 1$.

$f = 1$ on A $\qquad \int f$ is length of A

We're seeking a good length function for subsets of $\mathbb{R}$.

$$\ell : \mathcal{P}(\mathbb{R}) \longrightarrow [0, \infty]$$

1) $\ell([a,b]) = b - a$

2) If $A \subseteq \mathbb{R}$ and $c \in \mathbb{R}$ $\quad \ell(A+c) = \ell(A)$ (translation invariance)

3) If $A \subseteq \mathbb{R}$ and $r \in \mathbb{R}$ $\quad \ell(rA) = |r|\,\ell(A)$ (scaling covariance)

4) If $A \subseteq B$ $\quad \ell(A) \leq \ell(B)$. (monotonicity)

5) If A and B are disjoint then

$$\ell(A \cup B) = \ell(A) + \ell(B) \qquad \text{(finite additivity)}$$

(Exercise: $5) \Rightarrow 4)$)

$$\ell([-\varepsilon, \varepsilon]) = 2\varepsilon$$

$$\{0\} \subseteq [-\varepsilon, \varepsilon] \quad \forall\, \varepsilon > 0.$$

Consequence of 5) $\quad \ell(\phi) = \ell(\phi \cup \phi) = \ell(\phi) + \ell(\phi)$

so $\ell(\phi) = 0$ or ∞.

How long should $\mathbb{Z}$ be?

$$\mathbb{Z} = \bigcup_{n\in\mathbb{Z}} \{n\}$$

$$\ell(\mathbb{Z}) = \ell\Big(\bigcup_{n\in\mathbb{Z}}\{n\}\Big) = \sum_{n\in\mathbb{Z}} \ell(\{n\}) = \sum_{n\in\mathbb{Z}} 0 = 0.$$

a countable variation of 5) (finite additivity)

How long should $\mathbb{Q}$ be? Same argument suggests 0.

~~6)~~ 7) If $\{A_k\}_{k=1}^{\infty}$ are disjoint then

$$\ell\Big(\bigcup_{k=1}^{\infty} A_k\Big) = \sum_{k=1}^{\infty} \ell(A_k)$$

countable additivity

~~7)~~ 6) Given sets $\{A_k\}_{k=1}^{\infty}$ (not necessarily disjoint)

$$\ell\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} \ell(A_k)$$

countable subadditivity

Prop: If $f: \mathcal{P}(\mathbb{R}) \to [0,\infty]$ Then

f is countably additive iff it

is finitely additive and countably subadditive.

7) $\Longleftrightarrow$ 5), 6)