Spaces of bounded functions

X set, $X \neq \emptyset$

$$B(X) = \{ f: X \to \mathbb{R} : \exists M \geq 0 \text{ s.t. } |f(x)| \leq M \ \forall x \in X \}$$

$B(X)$ is a vector space.

Norm: $\|f\|_\infty = \sup_{x \in X} |f(x)|$ Exercise: this is a norm.

If $X = \mathbb{N}$ this is ℓ_∞.

$C[0,1] \subseteq B[0,1]$

↳ closed subspace

Convergence in $B(X)$ is

precisely uniform convergence.

$f_n \to f$ uniformly $\qquad f_n : X \to \mathbb{R}$

$$\forall \varepsilon > 0 \ \exists N \text{ s.t. } n \geq N \Rightarrow \underbrace{|f_n(x) - f(x)| \leq \varepsilon \ \ \forall x \in X}$$

$$\Leftrightarrow$$

$$\exists N \text{ s.t. } n \geq N \Rightarrow \|f_n - f\|_\infty \leq \varepsilon$$

$$\downarrow$$

$$\sup_{x \in X} |f_n(x) - f(x)|$$

Z

$\uparrow$ sequences ending all 0's $\subseteq \ell_2$

More generally if Y is a normed vector space

$$B(X,Y) = \{ f : X \to Y : \exists M \text{ with } \|f(x)\|_Y \leq M \ \ \forall x \in X \}$$

Again $B(X,Y)$ is a vector space and

$$\|f\|_\infty = \sup_{x\in X} \|f(x)\|_Y \quad \text{is a norm on } B(X,Y).$$

Prop: If Y is complete then so is $B(X,Y)$.
In particular, $B(X)$ is complete.

Pf: Suppose (f_n) is Cauchy in $B(X,Y)$.
So for all $\varepsilon > 0$ there exists N such that if $n,m \geq N$
then $\|f_n(x) - f_m(x)\|_Y < \varepsilon$ for all $x \in X$.
Hence, for each $x\in X$, $(f_n(x))$ is Cauchy in Y.
So for each $x \in X$, $f_n(x) \to f(x)$ for some $f(x) \in Y$.
Since Cauchy sequences are bounded there exists M
such that $\|f_n\|_\infty \leq M$ for all n.

Consequently $\| f_n(x) \|_Y \leq M$ for all n and all $x \in X$.

But then if $x \in X$, $\| f(x) \|_Y = \lim\limits_{n \to \infty} \| f_n(x) \| \leq M$ as well.

So $f \in B(X, Y)$.

To see that $f_n \to f$ in $B(X,Y)$ consider $\varepsilon > 0$.

Pick N such that if $n, m \geq N$, $\| f_n(x) - f_m(x) \|_Y \leq \varepsilon$

for all $x \in X$. Fix $n \geq N$. Then for each $x \in X$

$$\| f_n(x) - f(x) \|_Y = \lim_{m \to \infty} \| f_n(x) - f_m(x) \|_Y$$

$$\leq \varepsilon$$

since $\| f_n(x) - f_m(x) \|_Y \leq \varepsilon$ for m large enough.

That is, if $n \geq N$ then $\| f_n - f \|_\infty \leq \varepsilon$.

So $f_n \to f$.

We'd like a method for determining if series of functions converge uniformly.

Thm Weierstrass M-test

Suppose (f_n) is a sequence of functions from a set X to $\mathbb{R}$. Suppose moreover there exists constants $M_n \geq 0$ such that for all $x \in X$, $|f_n(x)| \leq M_n$.

If $\sum_{n=1}^{\infty} M_n$ converges then $\sum_{n=1}^{\infty} f_n$ converges uniformly (i.e. in $B(X)$) to a limit f.

Pf: Let $s_n = \sum_{k=1}^{n} f_k$. Then if $n > m$ and if $x \in X$ then

$$\left| s_n(x) - s_m(x) \right| = \left| \sum_{k=m+1}^{n} f_k(x) \right|$$

$$\leq \sum_{k=m+1}^{n} |f_k(x)|$$

$$\leq \sum_{k=m+1}^{n} M_k .$$

Hence, since the partial sums of the series $\sum_{k=1}^{\infty} M_k$ are

are Cauchy, so is $\underline{(s_n(x))}$ for each $x \in X$.

Hence $s_n(x) \to f(x)$ for some $f(x) \in \mathbb{R}$.

To see that $s_n \to f$ uniformly let $\varepsilon > 0$.

Pick N so that if $n > m \geq N$ then

$$|s_n(x) - s_m(x)| \leq \varepsilon \quad \text{for all } x \in X.$$

Taking a limit in m,

$$|s_n(x) - f(x)| = \lim_{m\to\infty} |s_n(x) - s_m(x)| \leq \varepsilon$$

if $n \geq N_0$ so $s_n \rightrightarrows f$ uniformly.

Thm (Weierstrass M-test, fancy)

Suppose (f_n) is a sequence in $B(X,Y)$ where Y is complete. If $\sum_{n=1}^{\infty} \|f_n\|_\infty$ converges then $\sum_{n=1}^{\infty} f_n$ converges uniformly to a limit $f \in B(X,Y)$.

In practice: you find M_n with $\|f_n\|_\infty \leq M_n$ and such that $\sum_{n=1}^{\infty} M_n$ converges. ($\Rightarrow \sum_{n=1}^{\infty} \|f_n\|_\infty$ converges as well)

Pf: Since Y is complete so is $B(X,Y)$.

Hence absolutely summable series in $B(X,Y)$ converge.

We have assumed $\sum_{n=1}^{\infty} \|f_n\|_\infty$ converges, which is precisely that $\sum_{n=1}^{\infty} f_n$ is absolutely convergent.

Application: power series.

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^4}{6!} - \cdots -$$

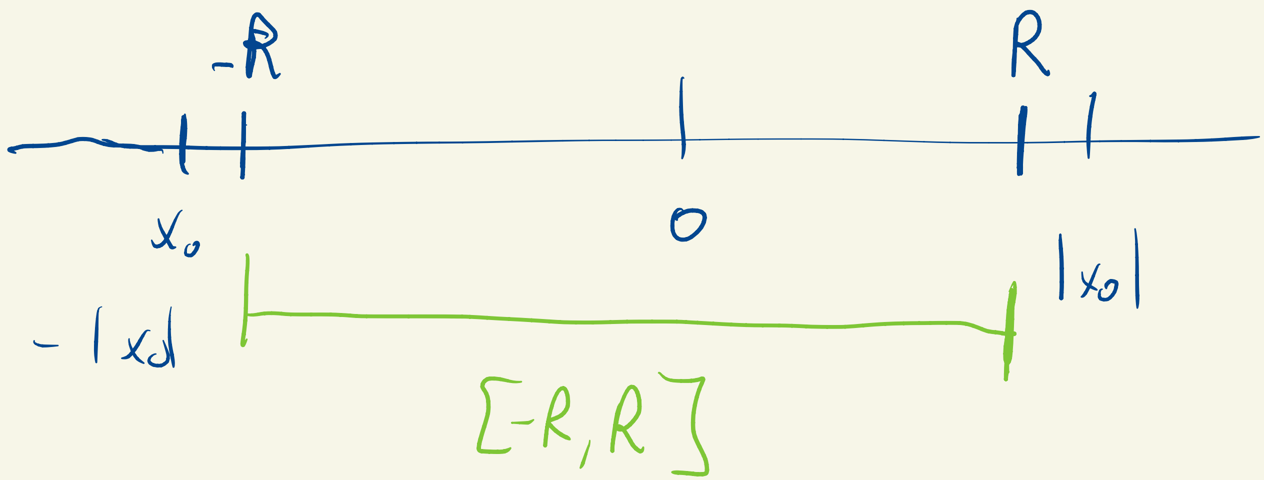
$$\cos'(x) = 0 \underbrace{- \frac{x}{1!} + \frac{x^3}{3!} - \frac{x^5}{5!} - \cdots -}_{-\sin(x)}$$

Given $\sum_{n=0}^{\infty} a_n x^n$ does it converge at some x's $\neq 0$.

If it does, call the resulting function $f(x)$.

Is $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$?

Suppose that for some $x_0 \neq 0$ that $\sum_{n=0}^{\infty} a_n x_0^n$ converges.



Claims

1) $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[-R, R]$

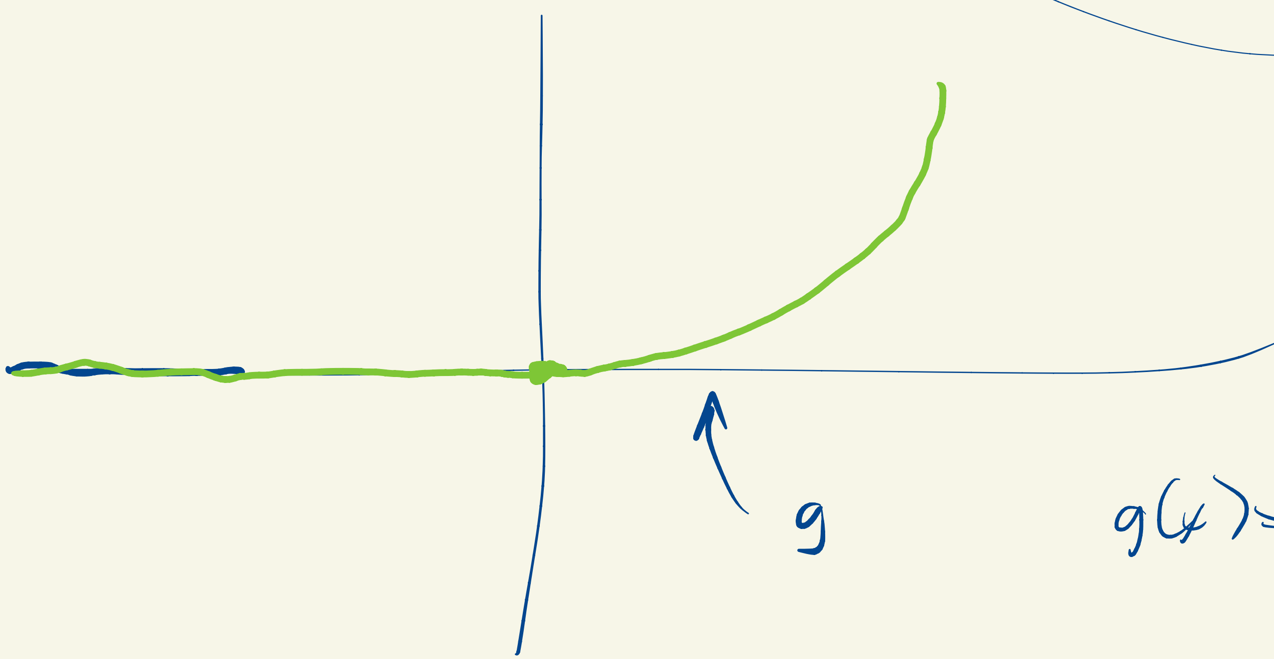
2) $\sum_{n=1}^{\infty} n a_n x^{n-1}$ converges uniformly on $[-R, R]$

3) $\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$ converges uniformly on $[-R, R]$

etc.

$f(x)$

$$a_0 = f(0)$$
$$a_1 = f'(0)$$
$$2a_2 = f''(0)$$
$$k!\,a_k = f^{(k)}(0)$$



$$g(x) = \begin{cases} 0 & x \le 0 \\ e^{-1/x} & x > 0 \end{cases}$$

$$g^{(k)}(0) = 0$$

g does not have a power series representation.

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a_0

$a_0 + a_1 x$

$a_0 + a_1 x + a_2 x^2$