Uniform convergence plays nicely with continuity and integration.

Prop: Suppose $f_n : X \to Y$ are all continuous at some $x_0 \in X$ and converge uniformly to a limit f. Then f is continuous at x_0.

"Uniform limit of cts functions is cts."

Pf: Let $\varepsilon > 0$. There exists N such that if $n \geq N$ then $d_Y(f(x), f_n(x)) < \varepsilon$ for all $x \in X$.

Since f_N is continuous at x_0 there exists $\delta > 0$ such that if $d_X(x_0, x) < \delta$, $d_Y(f_N(x_0), f_N(x)) < \varepsilon$.

Now, if $d_X(x_0, x) < \delta$

$$d_Y(f(x_0), f(x)) \leq d_Y(f(x_0), f_N(x_0)) + d_Y(f_N(x_0), f_N(x)) + d_Y(f_N(x), f(x))$$

$$< \varepsilon + \varepsilon + \varepsilon$$

$$= 3\varepsilon.$$

Hence f is continuous at x_0. $\square$

Integration:

First version:

$\Rightarrow$ Riemann int.

Prop: Suppose (f_n) is a sequence of continuous functions on $[a,b]$ and $f_n \to f$ uniformly. Then f is Riemann integrable and

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx.$$

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f$$

Pf: Since $f_n \to f$ uniformly, f is continuous and hence Riemann integrable.

To show $\lim\limits_{n\to\infty} \int_a^b f_n = \int_a^b f$, let $\varepsilon > 0$.

Pick N so that if $n \geq N$, $|f(x) - f_n(x)| < \varepsilon$ for all $x \in [a,b]$. Then, if $n \geq N$

$$\left| \int_a^b f_n - \int_a^b f \right| = \left| \int_a^b (f_n - f) \right|$$

$$\leq \int_a^b |f_n - f|$$

$$\leq \int_a^b \varepsilon$$

$$= (b-a)\,\varepsilon.$$

Hence $\int_a^b f_n \to \int_a^b f$.

$$f_n \rightrightarrows f$$

$$\Uparrow$$

$$f_n \to f$$

Differentiation:

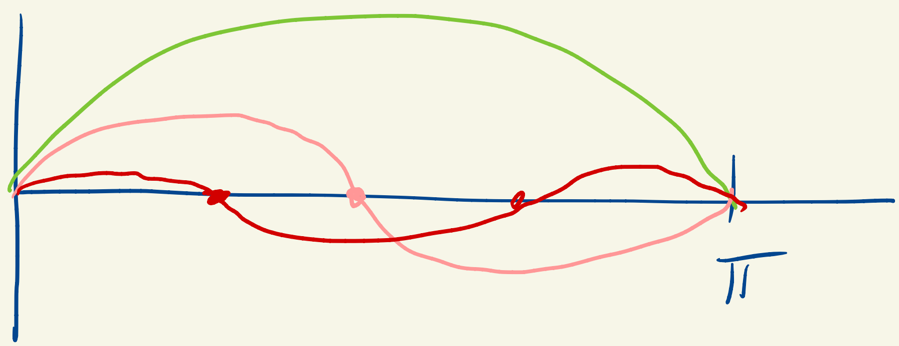
$f_n(x) = \frac{1}{n} x^n$

$f_n \to 0$ uniformly

$f_n'(1) = 1$.

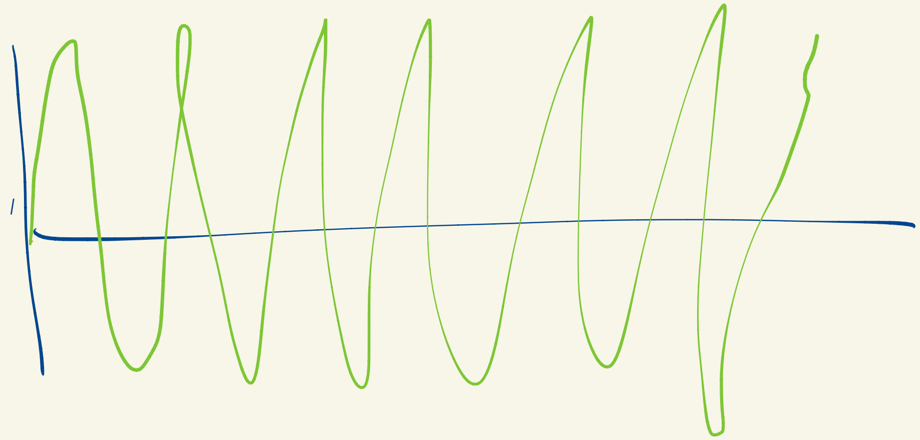
$f_n' \to 0'$ even pointwise

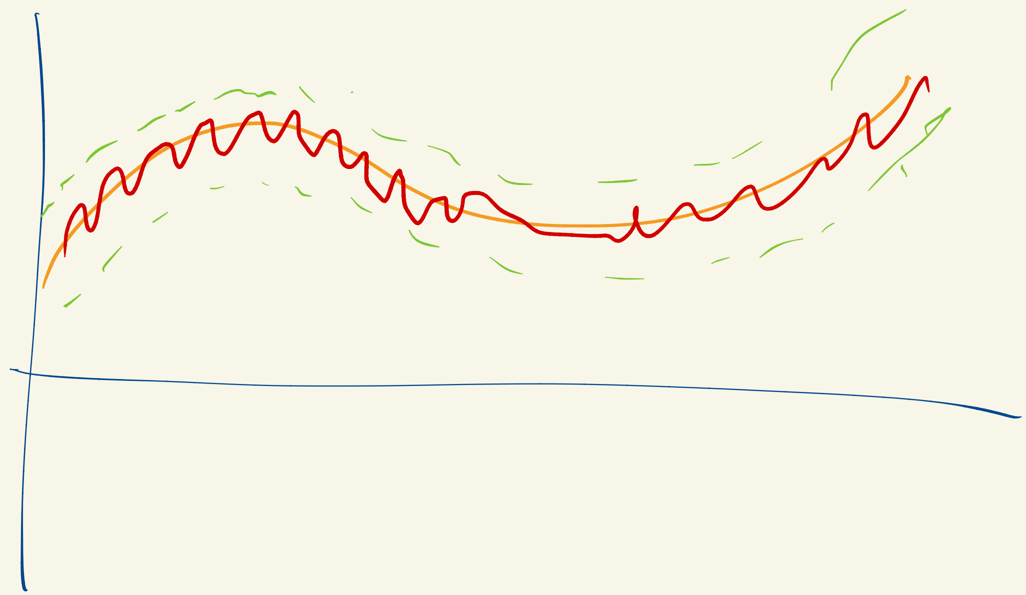
$f_n(x) = \frac{1}{n} \sin(nx)$ on $[0, \pi]$



$f_n \to 0$ uniformly

$f_n'(x) = \cos(nx)$





Prop: Suppose (f_n) is a sequence of functions on $[a,b]$ such that

1) Each f_n is cts. and differentiable on $[a,b]$
2) Each f_n' is continuous on $[a,b]$ $(*)$
3) $f_n' \to g$ uniformly for some g
4) $f_n(x_0) \to c$ for some $c \in \mathbb{R}$ and $x_0 \in [a,b]$.

Then there exists a differentiable function f on $[a,b]$ such that

1) $f_n \to f$ uniformly
2) f is differentiable and $f' = g$
3) $f(x_0) = c$

Pf: Observe that for each n

$$f_n(x) = f_n(x_0) + \int_{x_0}^{x} f_n'(s)\, ds.$$

This follows from the FTC using the fact that each f_n' is continuous.

Since $f_n' \to g$ uniformly and since $f_n(x_0) \to c$

$$f_n(x) \to c + \int_{x_0}^{x} g(s)\, ds.$$

Let $f(x) = c + \int_{x_0}^{x} g(s)\, ds$. We have just shown that $f_n \to f$ pointwise. Because g is a uniform limit of continuous functions it is continuous and the FTC then implies

$$f'(x) = g(x).$$

Evidently $f(x_0) = c$ and it remains to show $f_n \rightrightarrows f$ uniformly.

Observe that for any $x \in [a,b]$

$$\left| f_n(x) - f(x) \right| = \left| f_n(a) + \int_a^x f_n'(s)\,ds - \left(f(a) + \int_a^x f'(s)\,ds \right) \right|$$

$$\leq \left| f_n(a) - f(a) \right| + \left| \int_a^x (f_n' - \underbrace{f'}_{g})(s)\,ds \right|.$$

Let $\varepsilon > 0$. Pick N so that if $n \geq N$ then

$|f_n(a) - f(a)| < \varepsilon$ and such ~~that~~ that $|f_n'(x) - g(x)| < \varepsilon$

for all $x \in [a,b]$.

$$\left| \int_a^x (f_n' - f') \right| \leq \int_a^x |f_n' - f'| \leq \int_a^x \varepsilon = (x-a) \cdot \varepsilon \leq (b-a) \cdot \varepsilon$$

Hence, if $n \geq N$

$$|f_n(x) - f(x)| < \varepsilon + \int_a^x \varepsilon = \varepsilon + (x-a)\varepsilon \leq (1 + (b-a))\varepsilon,$$

for all $x \in [a,b]$.

$\square$