$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\frac{d}{dx} e^x = e^x$$

$$\frac{d}{dx} e^x \overset{?}{=} \frac{d}{dx} \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\overset{?}{=} \sum_{n=0}^{\infty} \frac{d}{dx} \frac{x^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{n x^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\int_0^1 e^x \, dx = \int_0^1 \sum_{n=0}^{\infty} \frac{x^n}{n!} \, dx$$

$$= \sum_{n=0}^{\infty} \int_0^1 \frac{x^n}{n!} \, dx$$

$$= \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} \Bigg|_0^1$$

$$= \sum_{n=0}^{\infty} \frac{1}{(n+1)!}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} - 1 = e - 1$$

$$f_N(x) = \sum_{k=0}^{N} \frac{1}{k!} x^k \qquad f(x) = e^x$$

$$f(x) = \lim_{N \to \infty} f_N(x) \qquad \forall x \in \mathbb{R}$$

Under such conditions generally is

$$\partial_x f(x) = \lim_{N \to \infty} \partial_x f_N(x)$$

is

$$\int_0^1 f(x)\,dx = \lim_{N \to \infty} \int_0^1 f_N(x)\,dx$$

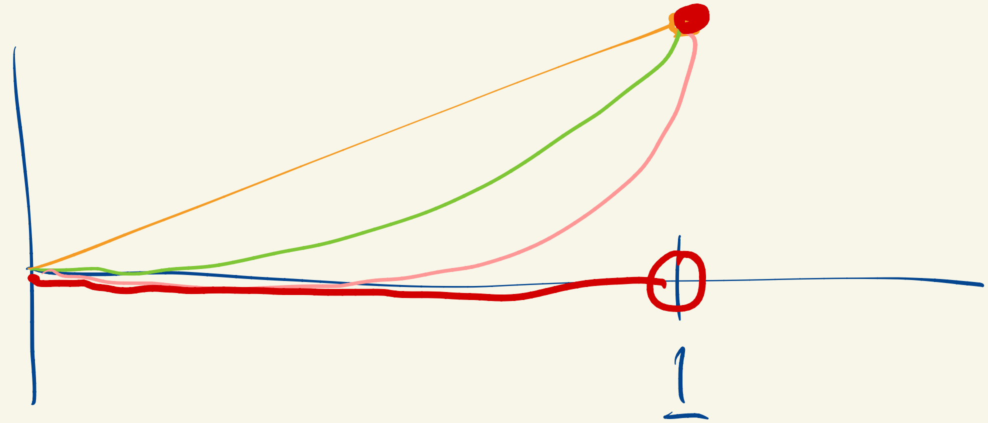
Def: Let (f_n) be a sequence of functions from a a set A to a metric space Y. We say the sequence converges pointwise to $f: A \to Y$ if for all $a \in A$, $f_n(a) \to f(a)$.

Grim News:

1) The pointwise limit of continuous functions need not be continuous.

$A = [0,1] \qquad Y = \mathbb{R}$

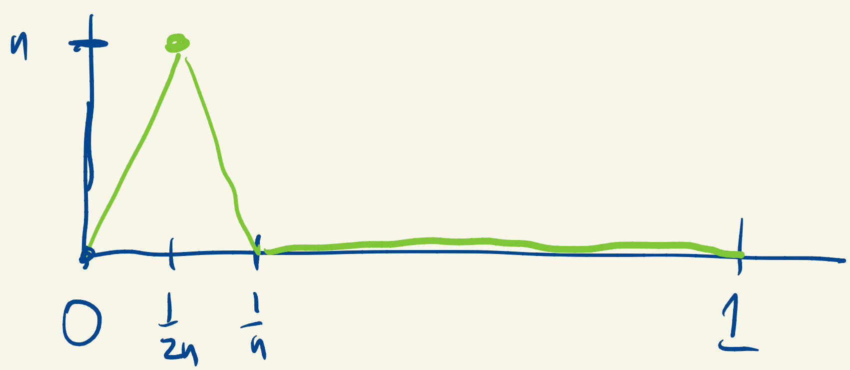
$$f_n(x) = x^n \qquad \lim_{n\to\infty} f_n(x) = \begin{cases} 0 & x \neq 1 \\ 1 & x = 1 \end{cases}$$



2) The pointwise limit of Riemann integrable functions need not be Riemann integrable. In the event that the limit is Riemann integrable it need not be

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx \qquad \left(f_n \to f \text{ pointwise on } [a,b] \right.$$

f_n



$[0,1]$

$f_n \to 0$ $\qquad$ $f_n(0) = 0$ for all n.

pointwise

$x \in (0,1]$ $\quad \exists N$ $\quad \frac{1}{N} < x$.

If $\quad n \geq N \quad f_n(x) = 0$

f

$\int_0^1 f_n(x)\,dx = \frac{1}{2}$ for all n

$\sin(\frac{1}{x})$ $\quad [\quad]$

$\int_0^1 f(x)\,dx = 0$

$\ne \quad x=0$

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases} \qquad 0 \leq x \leq 1$$

not Riemann integrable

(r_n) $\qquad f_n(x) = \begin{cases} 1 & x \in \{r_1, r_2, \dots, r_n\} \\ 0 & \text{otherwise} \end{cases}$

3) If $f_n \to f$ pointwise and the f_n's are differentiable it need not be the case that f is differentiable, In the event that it is, it need not be the case that $\lim\limits_{n\to\infty} f_n'(x) = f'(x)$, ☹

$[0,1]$ $\qquad f_n(x) = x^n$

$$g_n(x) = \frac{1}{n} x^n \qquad g_n \to 0 \text{ pointwise}$$

$$g_n'(x) = x^{n-1}$$

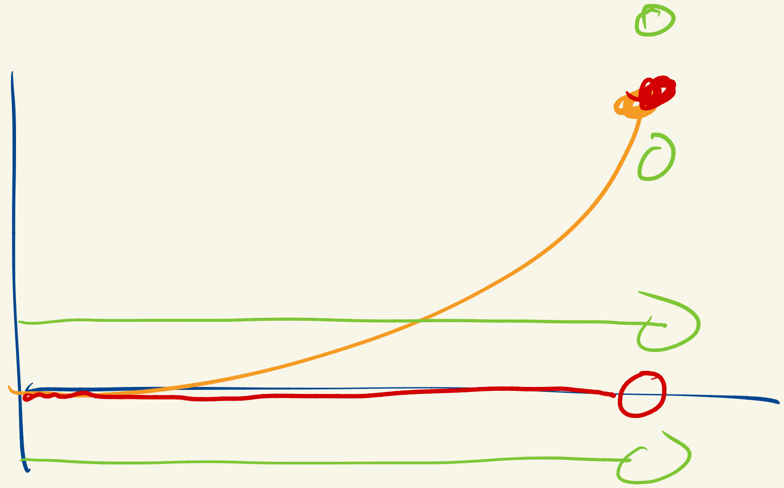
$$g_n'(1) = 1 \not\to 0.$$

Another notion of convergence of functions:

Def: Suppose (f_n) is a sequence of functions from a set A to a metric space Y. We say the sequence <u>converges uniformly</u> to a limit $f: A \to Y$ if for every $\varepsilon > 0$ there exists N such that for all $n \geq N$ and all $a \in A$,

$$d(f_n(a), f(a)) < \varepsilon.$$

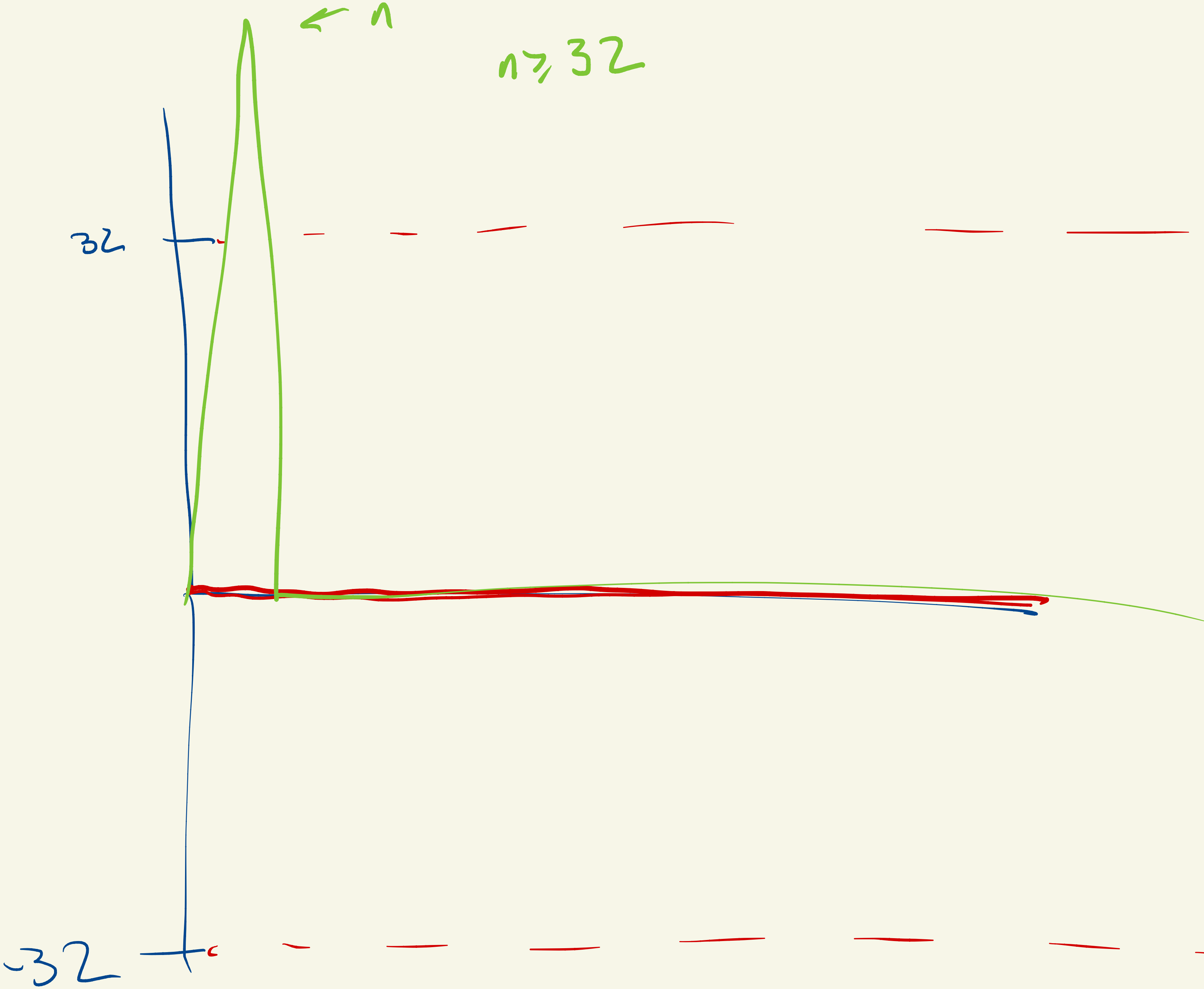
Note: uniform convergence implies pointwise convergence.



$\varepsilon = \frac{1}{4}$

$n \geq N$

An IVT argument shows the sequence does not converge uniformly.

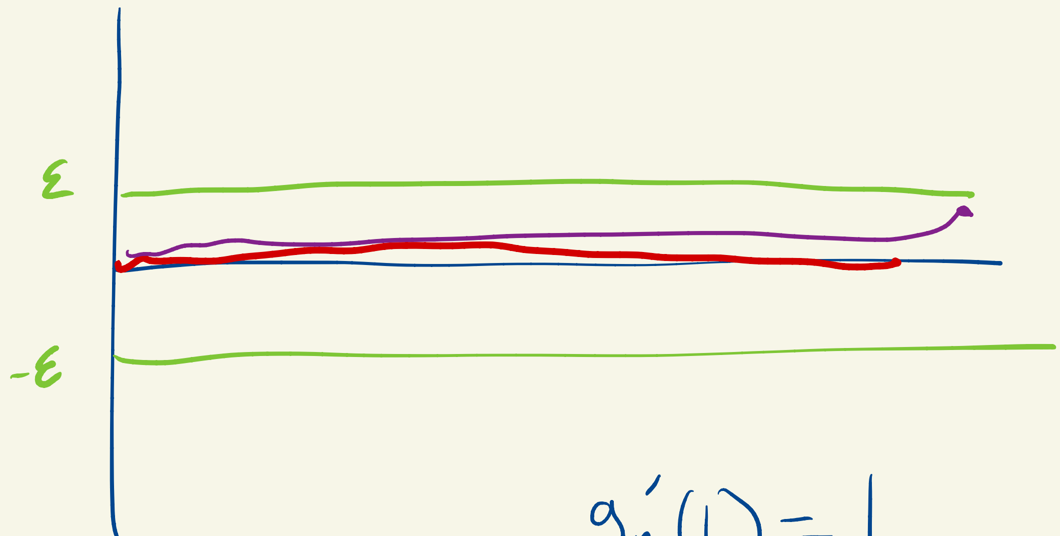

n
n ≥ 32
32
-32

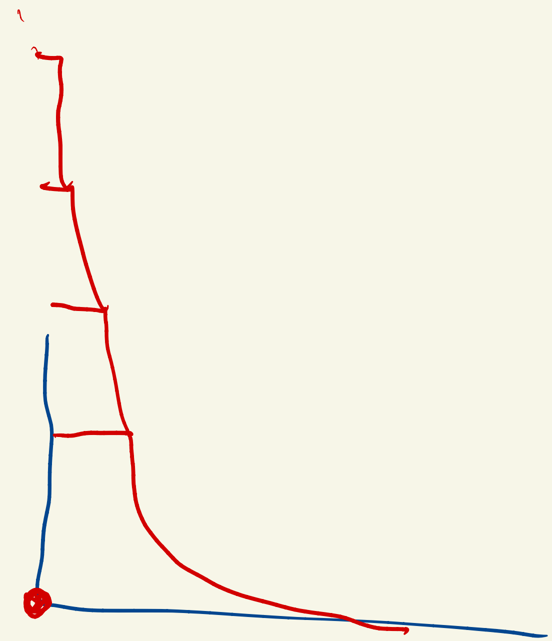
$g_n(x) = \frac{1}{n} x^n$

$g_n \to 0$ pointwise

$g_n \to 0$ uniformly.

N $\quad \frac{1}{N} < \varepsilon$





$g_n'(1) = 1$

$0'(1) = 0$

$f_n(x) = \frac{1}{x^{1+\frac{1}{n}}} = x^{-(1+\frac{1}{n})}$