$\|\cdot\|_1 \quad \|\cdot\|_2 \qquad\qquad c, C$

$$c\|x\|_1 \le \|x\|_2 \le C\|x\|_1 \quad \forall x \in X \ (x \neq 0)$$

Claim: All norms on $\mathbb{R}^n$ are equivalent.

(We'll show that an arbitrary norm $|||\cdot|||$ is equivalent to $\|\cdot\|_1$

ℓ_1.)

Ingredients

a) $|||\cdot|||$ is continuous w.r.t. ℓ_1.

b) $K = \{ x \in \mathbb{R}^n : \|x\|_1 = 1 \}$

is compact.

Want: $\exists\, c, C > 0$ s.t.

$$c\,\|x\|_1 \le |||x||| \le C\,\|x\|_1 \quad \text{for all } x \in \mathbb{R}^n \ \ x \neq 0.$$

$$\Leftrightarrow \quad c \le \frac{|||x|||}{\|x\|_1} \le C \quad \text{for all } x \in \mathbb{R}^n,\ x \neq 0.$$

$$\Leftrightarrow \quad c \le \frac{|||x|||}{\|x\|_1} \le C \quad \text{for all } x \in \mathbb{R}^n,\ \|x\|_1 = 1.$$

$$z \neq 0 \qquad x = \frac{z}{\|z\|_1} \qquad \|x\|_1 = 1$$

$$c \le \frac{|||\, z/\|z\|_1 \,|||}{\|\, z/\|z\|_1 \,\|_1} \le C$$

$$c \leq \frac{|||z|||}{\|z\|_1} \leq C$$

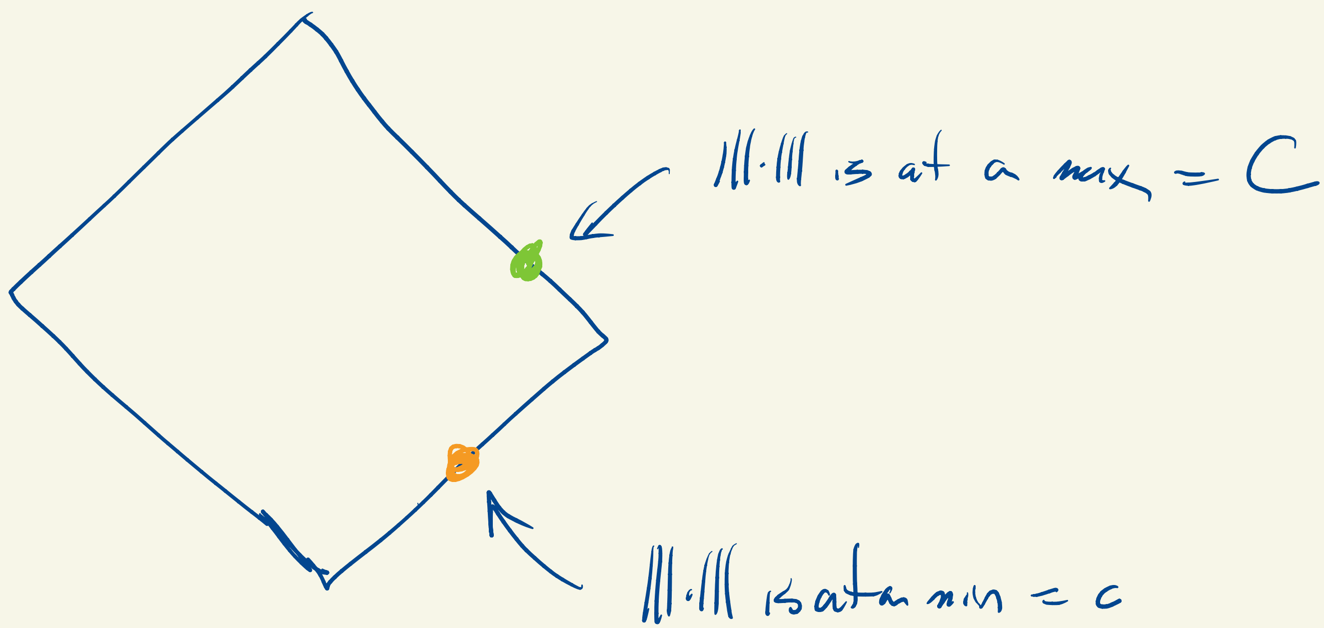
$\Longleftrightarrow \exists\, c, C > 0$

$$c \leq |||x||| \leq C \quad \text{for all } x \in K$$

The existence of $c, C \geq 0$ follows from continuity of $|||\cdot|||$ and compactness of K. (and $|||\cdot||| \geq 0$).

If $x \in K$ then $x \neq 0$, so $|||x||| > 0$.

Hence $c > 0$.



Continuity of $|||\cdot|||$.

Preliminary: Consider $x \in \mathbb{R}^n$. $e_{(k)} = (0, \dots, 0, 1, 0, \dots)$

slot k

$$C = \max_{k=1,\dots,n} |||e_{(k)}|||$$

$$x = \sum_{k=1}^{n} x_k e_{(k)}$$

$$|||x||| = |||\sum_{k=1}^{n} x_k e_{(k)}||| \le \sum_{k=1}^{n} |x_k| \, |||e_{(k)}|||$$

$$\le C \sum_{k=1}^{n} |x_k|$$

$$= C \, \|x\|_1$$

Continuity:

$$\big|\, |||x_2||| - |||x_1||| \,\big| \le |||x_2 - x_1||| \le C \, \|x_2 - x_1\|_1$$

$$\Downarrow$$

$$|f(x_2) - f(x_1)| \le \Lambda \, \|x_2 - x_1\|_1$$

$|||\cdot|||$ is Lip continuous.

Compactness of K.

Let (x_k) be a sequence in K, $x_k = (x_k(1), \dots, x_k(n))$.

Each $(x_k(\ell))$ is bounded in $\mathbb{R}$.

Hence we can extract a single subsequence (x_{k_j}) with $x_{k_j}(\ell) \to y(\ell)$ for some $y(\ell)$ for all $1 \le \ell \le n$.

$$\begin{array}{cccc} x_1(1) & x_1(2) & \cdots & x_1(n) \\ x_2(1) & x_2(2) & \cdots & x_2(n) \\ \vdots & \vdots & & \vdots \end{array}$$

Hence $x_{k_j} \xrightarrow{\ell_\infty} y$.

But then $x_{k_j} \xrightarrow{\ell_1} y$.

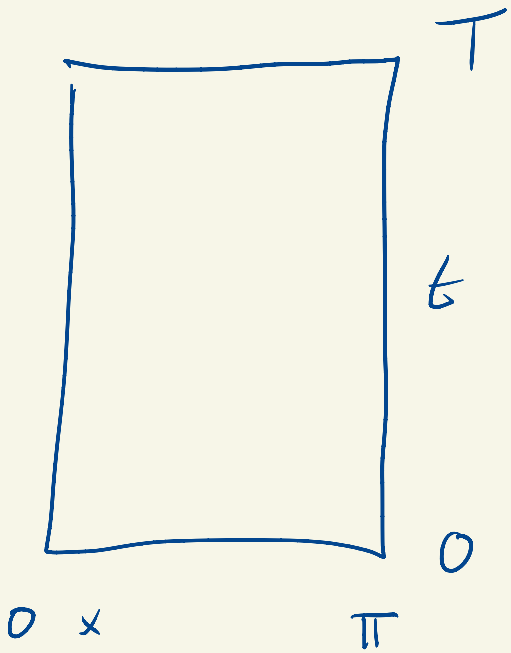
Is $y \in K$? Yep: $\|x_{k_j}\|_1 = 1$ for all j, and $\|\cdot\|_1$ is cts.

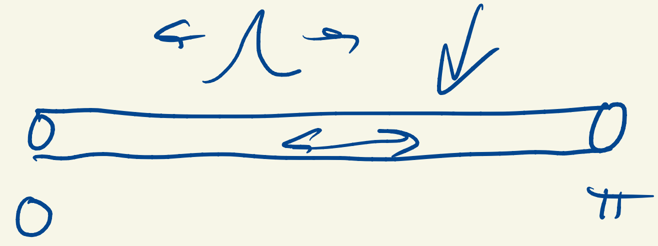
Exercise: A set $A \subseteq \mathbb{R}^n$ is compact w.r.t. ℓ_1 iff it is closed (w.r.t. ℓ_1 (or ℓ_2, or ℓ_∞)) and bounded.

Exercise: If V is a finite dimensional vector space all norms on V are equivalent.

[Show that if $T: \mathbb{R}^n \to V$ is a linear isomorphism then $x \longmapsto \|Tx\|_V$ is a norm on $\mathbb{R}^n$.

Heat Equation





$$u_t = u_{xx}$$

(u ~ temperature)

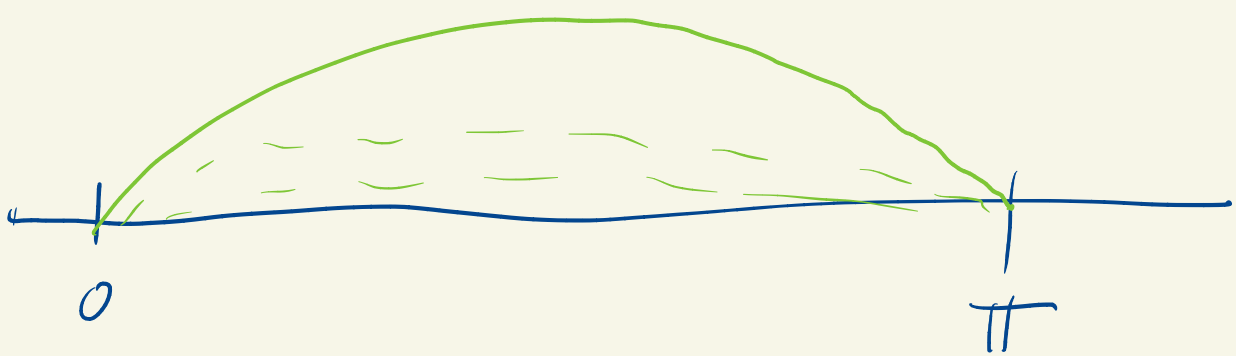
$$u\big|_{x=0} = 0 \qquad u\big|_{x=\pi} = 0$$

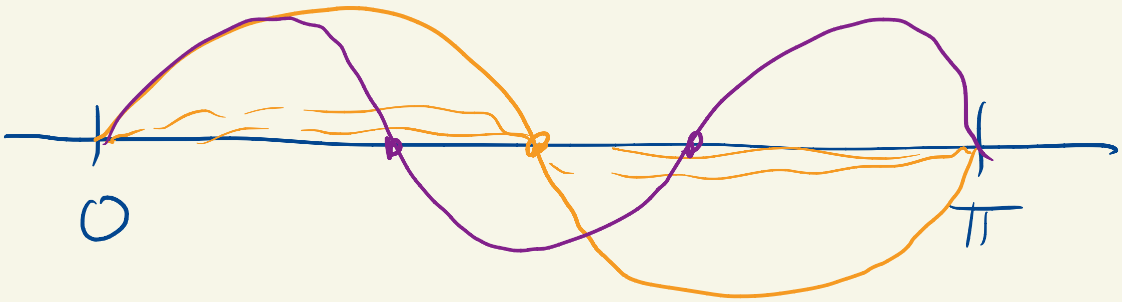
$$u(x,0) = u_0(x)$$

↳ initial temp

$$u_k(x,t) = e^{-k^2 t}\sin(kx)$$

$$e^{-t}$$





$k=2$

e^{-4t}

$$u = \sum_{k=1}^{N} c_k u_k$$

also solves the PDE ($u_t = u_{xx}$)

and $u|_{x=0} = u|_{x=\pi} = 0$.

$$u = \sum_{k=1}^{\infty} c_k u_k$$

does this solve the PDE?

In what sense?

For which coefficients c_k?

$$u(x,t) = \sum_{k=1}^{\infty} c_k u_k(x,t)$$

$$\partial_t u = \partial_t \sum_{k=1}^{\infty} c_k u_k(x,t)$$

$$\overset{?}{=} \sum_{k=1}^{\infty} c_k \partial_t u_k(x,t)$$

What kind of functions could I construct with

$$\sum_{k=1}^{\infty} a_k u_k(x,0)$$

Fact $\displaystyle\int_0^{\pi} \sin(kx)\sin(lx)\,dx = \begin{cases} 0 & \text{if } k \neq l \\ \frac{\pi}{2} & \text{if } k = l \end{cases}$

$$u_0 = \sum_{k=1}^{N} c_k \sin(kx)$$

$$\int_0^{\pi} u_0(x)\sin(lx)\,dx = \int_0^{\pi} \sum_{k=1}^{N} c_k \sin(kx)\sin(lx)\,dx$$

$$= \sum_{k=1}^{N} c_k \int_0^{\pi} \sin(kx)\sin(lx)\,dx$$

$$= c_l \frac{\pi}{2}$$

$$u_0 = \sum_{k=1}^{\infty} c_k \sin(kx)$$

---

u_0 cts

$$\frac{2}{\pi}\int_0^{\pi} u_0(x)\sin(\ell x)\,dx =: c_\ell$$
