$Z = \{$ sequences ending in a trail of 0's $\}$ $\qquad \ell_\infty$

$$Z \xhookrightarrow{f} \ell_1$$

$$z \longmapsto z$$

Claim: f is linear but not continuous.

$$z_n = (\underbrace{\tfrac{1}{n}, \tfrac{1}{n}, \dots, \tfrac{1}{n}}_{n}, 0, 0 \dots) \qquad \|z_n\| = \frac{1}{n}$$

$$z_n \longrightarrow 0$$

$$f(z_n) \xrightarrow{?} f(0) = (0, 0, \dots)$$

$$z_n \xrightarrow{\ell_1} 0 \text{ ? } \quad \text{Nope.} \quad \|z_n\|_1 = 1$$

Continuity of Linear Maps

Lemma: Translation is continuous on a normed vector space

$\left(T_{x_0}(x) = x + x_0, \ T_{x_0} \text{ is cts} \right).$

Lemma: Suppose $T: X \to Y$ is linear. Then

T is continuous if and only if it is continuous at 0.

Pf. Evidently, if T is continuous then it is continuous at 0.

Conversely, suppose T is continuous at 0 and that

$x_n \to x$ in X. (Want to show $T(x_n) \to T(x)$).

However, $x_n - x \to 0$ so

$T(x_n - x) \to T(0) = 0$ by continuity at 0.

But then $T(x_n) - T(x) \to 0$ and by continuity of translation $T(x_n) \to T(x)$.

We're going to characterize continuity at 0.

Def: A linear map $T: X \to Y$ is <u>bounded</u> if there exists $C > 0$ such that

$$\|T(x)\|_Y \leq C\|x\|_X \quad \text{for all } x \in X.$$

Prop: Suppose $T: X \to Y$ is linear. Then TFAE

1) T is bounded
2) $T(B_1^X(0))$ is a bounded subset of Y
3) T is continuous at 0.

Pf: $1) \Rightarrow 2)$

Suppose T is bounded with associated constant C.

Let $x \in B_1(0)$. Then $\|T(x)\|_Y \leq C\|x\|_X < C$.

Hence $T(B_1^X(0)) \subseteq B_C^Y(0)$.

$2) \Rightarrow 1)$

Suppose $T(B_1^X(0)) \subseteq B_C^Y(0)$.

Consider some $x \in X$, $x \neq 0$. Then

$\frac{x}{2\|x\|} \in B_1^X(0)$. So $T\left(\frac{x}{2\|x\|}\right) \in B_C^Y(0)$.

Hence $\left\|T\left(\frac{x}{2\|x\|_X}\right)\right\|_Y < C$.

But $\left\|T\left(\frac{x}{2\|x\|_X}\right)\right\|_Y = \left\|\frac{1}{2\|x\|_X} T(x)\right\|_Y = \frac{1}{2\|x\|_X}\|T(x)\|_Y$.

Hence $\|T(x)\|_Y < 2C\|x\|_X$. So T is bounded.

2) $\Rightarrow$ 3) Suppose $T(B_1^X(0))$ is bounded and hence contained in some $B_C^Y(0)$, $C > 0$.

To see that T is continuous let $\varepsilon > 0$.

Let $\delta = \varepsilon / C$.

If $\|x - 0\|_X < \delta$ then $x \in B_\delta^X(0)$

and $\frac{1}{\delta} x \in B_1^X(0)$. So $T(\frac{1}{\delta}x) \in B_c^Y(0)$

and hence, by linearity, $T(x) \in B_{\delta c}^Y(0) = B_\varepsilon^Y(0)$.

That is, $\|T(x) - T(0)\|_Y < \varepsilon$.

3) $\Rightarrow$ 2) Suppose T is continuous at 0. Taking $\varepsilon = 1$

we can find $\delta > 0$ so that if $\|x\|_X < \delta$,

$\|T(x)\|_Y < 1$.

Hence $T(B_\delta^X(0)) \subseteq B_1^Y(0)$ and

consequently $T(B_1^X(0)) \subseteq B_{1/\delta}^Y(0)$.

(Exercise $rT(B_\delta(0)) = T(B_{r\delta}(0))$,) $\square$

$$Z \xrightarrow{f} \ell_1$$

f is unbounded $\quad \|y_n\|_Z = 1$

$$y_n = (\underbrace{1, \dots, 1}_{n}, 0, \dots 0 \quad) \quad \|f(y_n)\|_{\ell_1} = n$$

Cor: Normed spaces X_1 and X_2 (being X with norms $|||\cdot|||_1$ and $|||\cdot|||_2$) have equivalent metrics if and only if there are constants c, C with

$$\boxed{c\,|||x|||_2 \le |||x|||_1} \le C\,|||x|||_2$$

for all $x \in X$.

$i : X_1 \to X_2$ is continuous iff there is $C > 0$

with $|||i(x)|||_2 \le C\,|||x|||_1$

$$(X, d_1) \xrightarrow{\text{cts}} (X, d_2)$$

$x \to x$

$$(X, d_2) \xrightarrow{\text{cts}} (X, d_1)$$

$$\|x\|_\infty \le \|x\|_1 \le n\,\|x\|_\infty \quad \text{on } \mathbb{R}^n$$

$T : X \to Y$, linear

T is cts $\iff \exists\ C$ s.t.

(*) $\qquad \|T(x)\|_Y \leq C\|x\|_X \quad \forall\ x \in X.$

If $\hat{C} \geq C$ then it also works

What is the best possible C that works?

If $x \neq 0$, then we can rewrite (*) as

$$\frac{\|Tx\|_Y}{\|x\|_X} \leq C$$

$$\underbrace{\sup_{\substack{x \in X \\ x \neq 0}} \frac{\|Tx\|_Y}{\|x\|_X}}$$ works as a best possible C.

C^*

$$\frac{\|Tx\|_Y}{\|x\|_X} \leq C^*$$

$$\|Tx\|_Y \leq C^* \|x\|_X$$

$\|T\|$, operator norm of T

$$\|T\| = \sup_{\substack{x \in X \\ x \neq 0}} \frac{\|Tx\|_Y}{\|x\|_X}$$

$B(X,Y)$ is the set of all continuous (bounded) linear maps from X to Y

Exercise: $\|\cdot\|$, operator norm, is a norm on $B(X,Y)$.

On $\mathbb{R}^n$

$$\|x\|_\infty \le \|x\|_1 \qquad \|x\|_1 \le n\|x\|_\infty$$

$$\|x\|_\infty \le \|x\|_2 \qquad \|x\|_2 \le \sqrt{n}\,\|x\|_\infty$$

$$\|x\|_2 \le \|x\|_1 \qquad \|x\|_1 \le \sqrt{n}\,\|x\|_2$$

On $\mathbb{R}^n$, the ℓ_1, ℓ_2 and ℓ_∞ norms are all equivalent

Claim: On $\mathbb{R}^n$, all norms are equivalent.

$$|||\cdot||| \qquad \qquad \|\cdot\|_1$$

$$e_{(k)} = (0, \dots, 1, 0 \dots 0)$$

k

$$|||x||| = |||\sum x_k e_{(k)}||| \leq \sum_k |x_k| \, |||e_{(k)}|||$$

$$C = \max_k |||e_{(k)}|||$$

$$|||x||| \leq \left(\sum_k |x_k|\right) C = C \|x\|_1$$