Cor: If $f: X \to Y$ is unif. cts then it takes Cauchy sequences to Cauchy sequences.

Pf: Suppose (x_n) is Cauchy in X.

Then $\{x_n : n \in \mathbb{N}\}$ is totally bounded as is

$$\{f(x_n) : n \in \mathbb{N}\}.$$

Hence $(f(x_n))$ is Cauchy.

Prop: Suppose X is compact and $f: X \to Y$ is cts.

Then f is uniformly continuous.

Pf: Suppose to the contrary that f is not uniformly continuous.

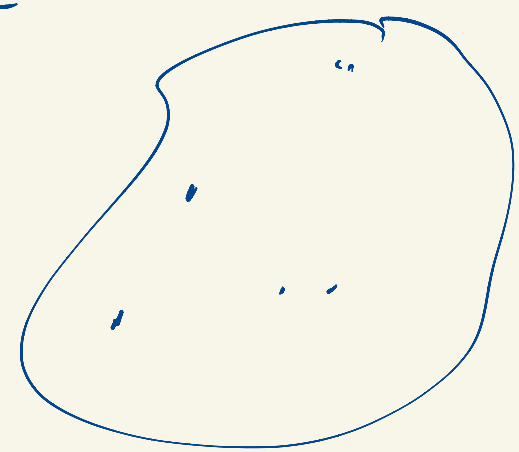
Then there exists $\varepsilon > 0$ such that for each $n \in \mathbb{N}$ we can find

a_n and b_n such that $d_X(a_n, b_n) < \frac{1}{n}$ but $d_Y(f(a_n), f(b_n)) \geq \varepsilon$.

Since X is compact we can extract a convergent

subsequence (a_{n_k}) with $a_{n_k} \to a$

for some $a \in X$.



We claim that $b_{n_k} \to a$ as well.

Observe $d_X(a, b_{n_k}) \leq d_X(a, a_{n_k}) + d_X(a_{n_k}, b_{n_k})$

$$\leq d_X(a, a_{n_k}) + \frac{1}{n_k}.$$

Since $d_X(a, a_{n_k}) \to 0$ and since $\frac{1}{n_k} \to 0$ we find $b_{n_k} \to a$. By continuity $f(a_{n_k}) \to f(a)$ and $f(b_{n_k}) \to f(a)$. But this contradicts the fact that $d_Y(f(a_{n_k}), f(b_{n_k})) \geq \varepsilon$ for all k.

$\square$

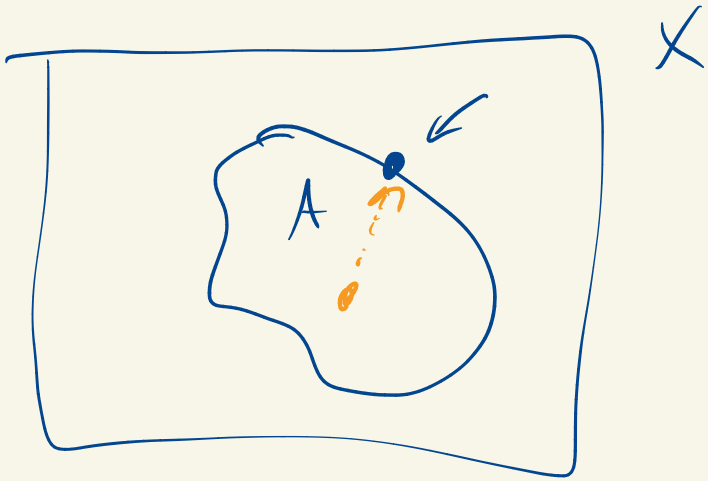
Task: $A \subseteq X$

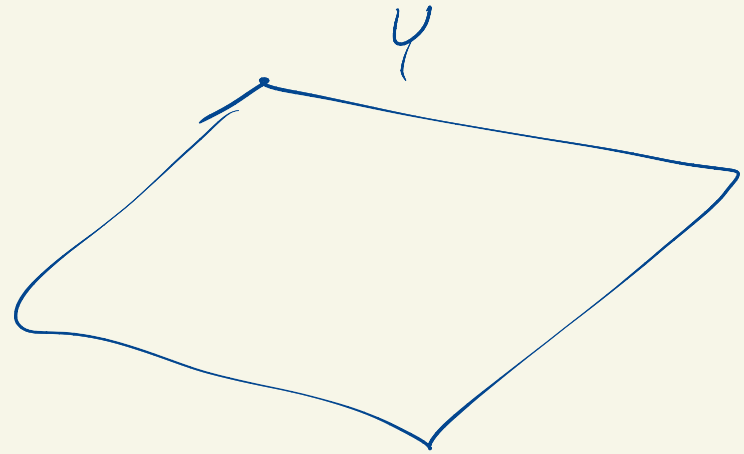
$f : A \to Y$, continuous

continuous

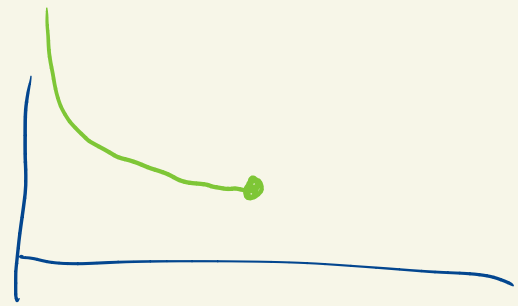
Can we construct $\bar{f} : \bar{A} \to Y$

such that $\bar{f}\big|_A = f$.

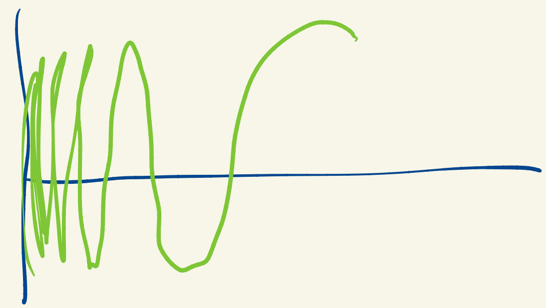




$f(x) = \frac{1}{x}$ on $(0,1]$



$f(x) = \sin\left(\frac{1}{x}\right)$ on $(0,1]$



Thm: Suppose $A \subseteq X$, $f: A \to Y$ is uniformly continuous, Y is complete and $\overline{A} = X$. Then there exists a unique continuous $\overline{f}: X \to Y$ such that $\overline{f}\big|_A = f$. Moreover, $\overline{f}$ is uniformly continuous.

Pf: Let $x \in X$. Let (a_n) be a sequence in A converging to x. Since (a_n) is Cauchy and since f is uniformly continuous, $(f(a_n))$ is Cauchy. Since Y is complete, $f(a_n) \to y$ for some $y \in Y$. We define $\overline{f}(x) = y$.

Is $\overline{f}$ well defined? Consider another sequence (b_n) with $b_n \to x$. Now construct the shuffled sequence

$(a_1, b_1, a_2, b_2, a_3, b_3, \ldots)$.

The shuffled sequence also converges to x and by the argument above $(f(a_1), f(b_1), f(a_2), f(b_2), \ldots)$

converges to some $\hat{y}$. But $(f(a_n))$ is a subsequence

and converges to y so $\hat{y} = y$. But then $f(b_k) \to y$

as well.

Note that if $x \in A$ we can use a constant sequence to find

$\bar{f}(x) = f(x)$, $\bar{f}$ is an extension of f.

To see that $\bar{f}$ is uniformly continuous let $\varepsilon > 0$.

Since f is uniformly continuous we can find $\delta > 0$

so that if $a, b \in A$ and $d(a,b) < \delta$ then $d(f(a), f(b)) < \frac{\varepsilon}{2}$.

Now consider $x, w \in X$ with $d(x,w) < \delta/3$.

Let (a_n) and (b_n) be sequences in A converging to

x and w respectively. Pick N so that if $n \geq N$

$d(a_n, x) < \frac{\delta}{3}$ and $d(b_n, w) < \frac{\delta}{3}$. Then

if $n \geq N$

$$d(a_n, b_n) \leq d(a_n, x) + d(x, w) + d(w, b_n)$$
$$< \frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3}.$$
$$= \delta.$$

So, if $n \geq N$, $d(f(a_n), f(b_n)) < \frac{\varepsilon}{2}$.

Now $f(a_n) \to \bar{f}(x)$ and $f(b_n) \to \bar{f}(w)$.

But $d(\bar{f}(x), \bar{f}(w)) = \underline{\lim_{n \to \infty} d(f(a_n), f(b_n))} \leq \frac{\varepsilon}{2} < \varepsilon$.

In particular $\bar{f}$ is continuous. The uniqueness of the extension follows from a HW exercise.

$\square$

Metrics are equivalent if they determine the same convergent sequences.

$$x_n \xrightarrow[d_1]{} x \iff x_n \xrightarrow[d_2]{} x$$

(*) If $x_n \xrightarrow[d_1]{} x$ then $x_n \xrightarrow[d_2]{} x$

(X, d_1) (X, d_2)

X_1 X_2

$id_{12} : X_1 \to X_2$

$id_{12}(x) = x$

(**) id_{12} is continuous

d_1 and d_2 are equivalent iff

id_{12} and $id_{21} = (id_{12})^{-1}$ are continuous.

In the context of normed vector spaces

$$(X, \|\cdot\|_1) \qquad (X, \|\cdot\|_2)$$

we have $id_{12} : X \to X$.

I claim id_{12} is linear.

$$id_{12}(x_1 + x_2) = id_{12}(x_1) + id_{12}(x_2)$$
$$id_{12}(cx) = c\, id_{12}(x).$$

$$id_{12}(x_1+x_2) = x_1+x_2 = id_{12}(x_1) + id_{12}(x_2)$$

$$id_{12}(cx) = cx = c\, id_{12}(x)$$

Are linear maps always continuous?

No.

$$P[0,1] \xrightarrow{d} P[0,1] \qquad \|\cdot\|_\infty$$

$$p_n(x) = \frac{1}{n}x^n \qquad p_n \to 0$$

$$d(p_n)(x) = x^{n-1}$$

↳ if $x=1$ this is 1.

$$\| \qquad \|_\infty \geq 1 \quad \text{for all } n.$$

$$p_n \to 0 \qquad \|d(p_n)\|_\infty \geq 1. \quad \text{So } d(p_n) \not\to 0.$$