Cauchy sequence in A with a convergent subsequence (in A) and hence converges in A.

Upshot:

Thm: A subset $A \subseteq X$ is compact iff it is $\underbrace{\text{complete}}$ and $\underbrace{\text{totally bounded}}$.

$X = \mathbb{R}$ $\quad$ $\mathbb{R}$: complete $\Leftrightarrow$ closed $\quad$ $\mathbb{R}$: totally bounded $\Leftrightarrow$ bounded

$\mathbb{R}$: compact $\Leftrightarrow$ closed + bounded

Q: Given a compact metric space, what are the compact subsets?

X : compact

$A \subseteq X$ Since X is totally bounded, so is A.

A is complete $\Leftrightarrow$ it is closed.

Prop: If X is compact and $A \subseteq X$ then A is compact $\Leftrightarrow$ A is closed.

Continuity need not preserve completeness nor total boundedness

$f(x) = \frac{1}{x}$ $\quad (0,1]$ $\quad f((0,1])$ is not bounded

$\mathbb{R} \longrightarrow (-1,1)$

The combination of the two, compactness, is preserved by continuity.

Prop: Suppose $f: X \to Y$ is continuous and $K \subseteq X$ is compact. Then $f(K)$ is compact as well.

Pf: Let (y_n) be a sequence in $f(K)$.

For each n we can pick $x_n \in K$ with $f(x_n) = y_n$.
Since K is compact we can extract a subsequence x_{n_j} converging to some $x \in K$. By continuity,

$$f(x_{n_j}) \to f(x) \in f(K).$$

That is $y_{n_j} \to f(x) \in f(K)$.

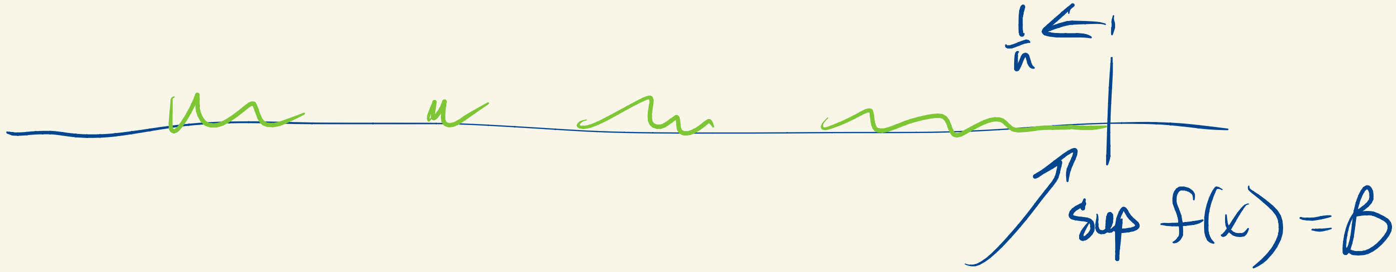
Cor: EVT (extreme value theorem)

Suppose X is compact and nonempty and $f: X \to \mathbb{R}$ is continuous.

Then there exist x_m and x_M in X such that

for all $x \in X$, $\quad f(x_m) \leq f(x) \leq f(x_M)$.

Pf: Since X is compact, $f(X)$ is a compact subset of $\mathbb{R}$ and is hence bounded and in particular bounded above. It is nonempty as X is. Let $B = \sup f(X)$. There is a sequence b_n in $f(X)$ converging to B. Since $f(X)$ is compact it is closed and hence $B \in f(X)$. Hence there exists $x_M \in X$ with $f(x_M) = B$. $\square$



$$C[0,1] = \{ f : [0,1] \to \mathbb{R} : f \text{ is cts.} \}$$

$$\|f\|_\infty = \sup_{x \in [0,1]} |f(x)| = \max_{x \in [0,1]} |f(x)|$$

If X is compact we can define a similar space

$$C(X) = \{ f : X \to \mathbb{R} : f \text{ is continuous} \}$$

$$\|f\|_\infty = \max_{x \in X} |f(x)|$$ (this is well defined because X is compact)

Exercise: $\|\cdot\|_\infty$ is a norm on $C(X)$.

Q: What subsets of $C(X)$ are compact?

$C[0,1]$

Topological compactness:

A space X is topologically compact if whenever $\{U_\alpha\}$ is a collection of open sets in X with $\bigcup U_\alpha = X$ then there is a finite subcollection $U_{\alpha_1}, \ldots, U_{\alpha_n}$ with $\bigcup_{k=1}^{n} U_{\alpha_k} = X$.

Equivalently, X is topologically compact if whenever

$\{F_\alpha\}$ is a collection of closed sets in X with the finite intersection property (i.e. any nonempty finite collection of F_α's has nonempty intersection) then

$$\boxed{\bigcap F_\alpha \neq \phi.} \qquad \left(\bigcap F_\alpha\right)^c \neq \phi^c$$

$$\bigcup (F_\alpha)^c \neq X$$

(Exercise: use DeMorgan's laws to show these are equivalent).

Compactness and topological compactness are the same.

(See text)

Uniform Continuity

Def A function $f: X \to Y$ is uniformly continuous if for every $\varepsilon > 0$ there is $\delta > 0$ so that if $x_1, x_2 \in X$ and $d(x_1, x_2) < \delta$ then $d(f(x_1), f(x_2)) < \varepsilon$.

One δ works everywhere

E.g. sin is uniformly cts.

it's Lip. with Lip. const 1.

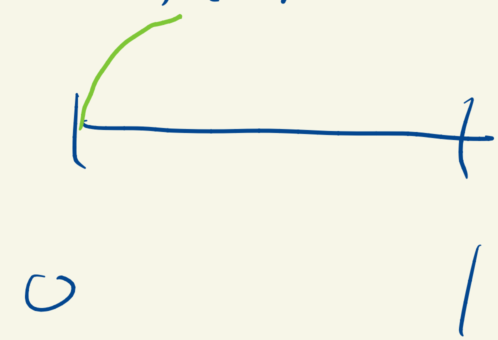
$$|\sin(x_1) - \sin(x_2)| \leq |x_1 - x_2|$$

More generally, if f is Lip. cts. with Lip. const K then f is unif. continuous.

$$d(f(x_1), f(x_2)) \leq K\, d(x_1, x_2).$$

Given $\varepsilon > 0$, pick $\delta = \varepsilon / K$.

$f(x) = \sqrt{x}$



$$\frac{f(x) - f(0)}{|x - 0|} \leq K$$

$$\frac{\sqrt{x} - \sqrt{0}}{x} = \frac{1}{\sqrt{x}} \to \infty \text{ as } x \to 0.$$

e.g. $f(x) = x^2$ $\quad f: \mathbb{R} \to \mathbb{R}$

this is not uniformly cts.

Def A function $f: X \to Y$ is uniformly continuous if for every $\varepsilon > 0$ there is $\delta > 0$ so that if $x_1, x_2 \in X$ and $d(x_1, x_2) < \delta$ then $d(f(x_1), f(x_2)) < \varepsilon$.

There is a bad $\varepsilon_0 > 0$ that for all $\delta > 0$ there exist unfortunate x_1 and x_2 such that $d(x_1, x_2) < \delta$ but $d(f(x_1), f(x_2)) \geq \varepsilon_0$.

$x > 0 \qquad h > 0$

$x_1 = x$

$x_2 = x + h$

$$|f(x_2) - f(x_1)| = |x_2^2 - x_1^2|$$

$$= |(x+h)^2 - x^2|$$

$$= 2xh + h^2$$

$$> 2xh$$

$\varepsilon_0 = 1 \qquad \delta > 0$

Pick $h < \delta$. Pick $x > \dfrac{1}{2h}$

$$|f(x_2) - f(x_1)| > 2xh > 1$$

$$|x_2 - x_1| = h < \delta$$

$\sin(1/x)$ on $(0,1]$

Equivalent formulation:

$\forall \varepsilon > 0$ there exists $\delta > 0$ such that

for all $x \in X$ $\quad f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$.

Exercise: show this is equivalent

Prop: Suppose $f: X \to Y$ is uniformly (!) continuous.
If $A \subseteq X$ is totally bounded then so is $f(A)$.

Pf: Suppose $A \subseteq X$ is totally bounded. Let $\varepsilon > 0$ and find $\delta > 0$ such that for all $x \in X$, $f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$.

Since A is totally bounded there exists a δ-net $a_1, \ldots, a_n$ for A. So $A \subseteq \bigcup_{k=1}^{n} B_\delta(x_k)$.

But then

$$f(A) \subseteq f\left(\bigcup_{k=1}^{n} B_\delta(x_k)\right)$$

$$= \bigcup_{k=1}^{n} f\left(B_\delta(x_k)\right)$$

$$\subseteq \bigcup_{k=1}^{n} B_\varepsilon\left(f(x_k)\right).$$

Hence $f(x_1), \ldots, f(x_n)$ is an ε-net for $f(A)$.

$\square$