Thm: Suppose X is complete. Then $A \subseteq X$ is complete if and only if it is closed.

Pf: Suppose $A \subseteq X$ is complete. Suppose $\{a_n\}$ is a sequence in A converging to some $x \in X$. We wish to show $x \in A$. Now $\{a_n\}$ is convergent and hence Cauchy in X and therefore Cauchy in A. Since A is complete, the sequence converges to some a in A. But convergence in A implies convergence in X. Since limits of sequences in X are unique, $x = a \in A$.

Suppose A is closed and let $\{a_n\}$ be a Cauchy sequence in A. Then it is also Cauchy in X. But then it converges in X to a limit x. Since A is closed it contains the limits of its convergent sequences and $x \in A$. But then $\{a_n\}$ converges in A to x as well. $\square$

Def: A <u>Banach space</u> is a complete normed linear space.

e.g. $\mathbb{R}$

$(\mathbb{R}^2, l_2)$ $\qquad$ $\underline{l_1, l_2, l_\infty}$

l_2 $\qquad$ (l_1, l_∞, c_0)

$(\underline{C[0,1]}, L_2)$ not complete

Given a normed vector space there is a standard tool for showing that it is complete.

$$\sum_{k=1}^{\infty} x_k \qquad \sum_{k=1}^{\infty} |x_k|$$

$$(1) \qquad\qquad (2)$$

we say (1) is absolutely convergent if (2) is.

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$$ is convergent but not absolutely convergent.

From undergrad analysis: absolutely convergent series converge.

Thm: A normed linear space X is a Banach space if and only if every absolutely convergent series in the space converges.

Def: A series $\sum_{k=1}^{\infty} x_k$ is absolutely convergent if

if $$\sum_{k=1}^{\infty} \|x_k\|$$ converges.

Pf: Suppose X is complete and $\sum_{k=1}^{\infty} x_k$ is absolutely convergent. We wish to show that the series converges, i.e. the sequence of partial sums converge.

Let $s_N = \sum_{k=1}^{N} x_k$. Then if $N < M$

$$\| s_N - s_M \| = \left\| \sum_{k=N+1}^{M} x_k \right\| \leq \sum_{k=N+1}^{M} \| x_k \|.$$

Since the series $\sum_{k=1}^{\infty} \|x_k\|$ converges, its sequence of partial sums is convergent and hence Cauchy. But then so is $\{s_N\}$. Since

X is complete, the sequence $\{S_N\}$ converges.

That is $\sum_{k=1}^{\infty} x_k$ converge.

Conversely, suppose absolutely convergent series in X converge. Let $\{x_k\}$ be a Cauchy sequence in X.

Find N_1 so that if $n, m \geq N_1$ then $\|x_n - x_m\| \leq \frac{1}{2}$.

Find $N_2 > N$ so that if $n, m \geq N_2$ then $\|x_n - x_m\| \leq \frac{1}{2^2}$

Continuing inductively, choose $N_{k+1} > N_k$ such that if $n, m \geq N_k$, $\|x_n - x_m\| \leq \frac{1}{2^{k+1}}$.

Consider the subsequence $\{x_{N_k}\}_{k=1}^{\infty}$.

Observe

$$x_{N_k} = x_{N_1} + (x_{N_2} - x_{N_1}) + \cdots + (x_{N_k} - x_{N_{k-1}}).$$

Moreover $\sum_{j=2}^{k} \| x_{N_j} - x_{N_{j-1}} \| \leq \sum_{j=2}^{k} \frac{1}{2^{j-1}}$.

By the comparison test, $\sum_{j=2}^{\infty} \| x_{N_j} - x_{N_{j-1}} \|$ is convergent

and hence $\sum_{j=2}^{\infty} (x_{N_j} - x_{N_{j-1}})$ is convergent also, as is $\sum_{j=0}^{\infty} x_{N_j} - x_{N_{j-1}}$

But then $\{x_{N_k}\}$ converge as well.

But then $\{x_{k}\}$ is a Cauchy sequence with a convergent subsequence and it converges.

Def: A set A in a metric space is compact if every sequence in A has a subsequence that converges to a point in A.

Lemma: Suppose $A \subseteq X$ is complete and totally bounded. Then it is compact.

Pf: Since A is totally bounded every sequence in A has a Cauchy subsequence which converges in A by the completeness of A.

Lemma: Suppose $A \subseteq X$ is compact. Then A is totally bounded.

Pf: Let $\{a_n\}$ be a sequence in A.

Then it has a subsequence which converges ~~to~~ some point in A. This subsequence is Cauchy and hence A is totally bounded.

Lemma: Suppose $A \subseteq X$ is compact. Then A is complete.

Pf. Let $\{a_n\}$ be a Cauchy sequence in A.

Since A is compact the sequence has a convergent subsequence converging to a point in A, So it is a

Cauchy sequence in A with a convergent subsequence (in A) and hence converges in A.

Upshot:

Thm: A subset $A \subseteq X$ is compact iff it is $\underbrace{\text{complete}}$ and $\underbrace{\text{totally bounded}}$.

$$(x_1, x_2, x_3, \ldots)$$

$$\underbrace{x_1}_{y_1} + \underbrace{(x_2 - x_1)}_{y_2} + \underbrace{(x_3 - x_2)}_{y_3} + \underbrace{(x_4 - x_3)}_{y_4}$$

$$y_1 = x_1$$

$$y_1 + y_2 = x_2$$

$$y_1 + y_2 + y_3 = x_3$$

$$\sum_{k=1}^{\infty} y_k$$

$$S_N = \sum_{k=1}^{N} y_k$$

$$S_N = x_N$$

$$\sum_{k=1}^{\infty} \|y_k\| = \sum_{k=1}^{\infty} \|x_k - x_{k-1}\|$$

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$$

$$\frac{1}{k} - \frac{(-1)}{k+1} = \frac{k+1+k}{k(k+1)} = \frac{2k+1}{k^2+1}$$

$$\geq \frac{c\,.}{k}$$

$$\frac{2+1/k}{k+1}$$

$$\geq \frac{2}{k+1}$$