Continuing inductively we can find a sequence (a_k) with each a_k in A and $d(a_k, a_\ell) \geq \varepsilon$ if $k \neq \ell$.

No subsequence can be Cauchy, for any cauchy subsequence would contain two terms at distance no more than $\varepsilon/2$ from each other.

Cor: (Bolzano-Weierstrass Thm)

Every bounded sequence in $\mathbb{R}$ has a convergent subsequence

Pf: Suppose (x_n) is a sequence in $[-R, R]$ for some $R > 0$. Last class we showed that $[-R, R]$

is totally bounded and hence the theorem above shows the sequence has a Cauchy subsequence. Hence, by the completeness of $\mathbb{R}$ the subsequence converges. $\square$

This is a 1 - 2 punch

get Cauchy

use completeness.

Def. A metric space X is complete if every Cauchy sequence in X converges.

Examples: 1) $\mathbb{R}$

2) $\mathbb{R}^2$ with l_1 norm?

Suppose $z_n = (x_n, y_n)$ is a Cauchy sequence.

Observe then $|x_n - x_m| \leq \overbrace{\|z_n - z_m\|_1}^{|x_n - x_m| + |y_n - y_m|}$.

Construct a candidate limit (x, y).

Given $\varepsilon > 0$ we can find N so if $n, m \geq N$ then $\|z_n - z_m\|_1 < \varepsilon$.

But then if $n, m \geq N$, $|x_n - x_m| \leq \|z_n - z_m\|_1 < \varepsilon$.

So (x_n) is Cauchy and converges to a limit x_0.

Similarly $y_n \to y_0$

Next: show $z_n \to (x,y) = z$

$$\| z - z_n \|_1 = \underline{|x - x_n|} + |y - y_n|$$

$$|x - x_n| \to 0$$

$$|y - y_n| \to 0$$

$w_n \to w$ in X

$\Longleftrightarrow$

$\underline{d(w, w_n)} \to 0$

$$|x - \cancel{x}x_n| + |y - y_n| \to 0 + 0$$

$$\| z - z_n \|_1 \to 0$$

$$d_{l_1}(z, z_n) \to 0$$

$$z_n \to z$$

Then $\lim_{n\to\infty} \|z_n - z\|_1 = 0$.

Then $\|z_n - z\|_1 \to 0$.

Procedure: a) exhibit a candidate limit.

a') Show the candidate is in the space under consideration.

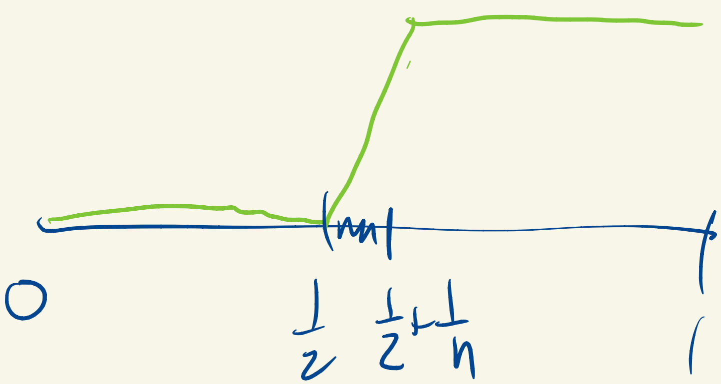
b) prove convergence to the candidate

We'll see in the near future that ℓ_2 is complete,

You'll show (HW) that ℓ_1, ℓ_∞, c_0 are all complete.

We will show $(C[0,1], L_\infty)$ is complete.

But: we have already seen that $(C[0,1], L_1)$ is not complete.



(X, d_1) $\quad (x_n) \xrightarrow{d_1} x$

(Y, d_2) $\quad (x_n) \xrightarrow{d_2} y$

$x \neq y$

Let's show l_2 is complete.

Consider a Cauchy sequence (x_n) in l_2.

Each x_n is a sequence of real numbers; let

$x_n(k)$ denote the k^{th} term of the sequence x_n.

So $x_n = (x_n(1), x_n(2), x_n(3), \ldots)$

We need a candidate limit.

$x = (x(1), x(2), x(3), \ldots)$

For each k

$$|x_n(k) - x_m(k)|^2 \leq \sum_{k=1}^{\infty} |x_n(k) - x_m(k)|^2 = \|x_n - x_m\|_2^2.$$

That is, $|x_n(k) - x_m(k)| \leq \|x_n - x_m\|_2$.

Since (x_n) is Cauchy, each sequence $(x_n(k))$ is a Cauchy sequence in $\mathbb{R}$ and converges to a limit $x(k)$.

Candidate: $x = (x(k))$

Is $x \in l_2$? Does $x_n \to x$ in l_2?

To see that $x \in l_2$ observe that for each K

$$\sum_{k=1}^{\overset{\neq \infty}{K}} |x(k)|^2 = \lim_{n\to\infty} \sum_{k=1}^{\overset{\neq \infty}{K}} |x_n(k)|^2 .$$

$$\left[\; x_n(k) \to x(k) \Rightarrow |x_n(k)|^2 \to |x(k)|^2 \; \right]$$

$$x_n(k) \to x(k) \Rightarrow \sum_{k=1}^{\infty} |x_n(k)|^2 \to \sum_{k=1}^{\infty} |x(k)|^2$$

$$\lim_{n\to\infty} \lim_{K\to\infty} \sum_{k=1}^{K} |x_n(k)|^2 \text{ vs. } \lim_{K\to\infty} \lim_{n\to\infty} \sum_{k=1}^{K} |x_n(k)|^2$$

$$\sum_{k=1}^{K} |x(k)|^2 = \lim_{n\to\infty} \sum_{k=1}^{K} |x_n(k)|^2.$$

$$\leq \limsup_{n\to\infty} \|x_n\|_2^2.$$

$y_n \leq z_n \quad \forall n$

$$\limsup_{n\to\infty} y_n \leq \limsup_{n\to\infty} z_n$$

$\Downarrow$

$\lim_{n\to\infty} y_n$

Since (x_n) is Cauchy in l_2 it is bounded in l_2.

Let $M = \limsup_{n\to\infty} \|x_n\|_2$,

So $M < \infty$.

We've shown there exists $M > 0$ such that

$$\sum_{k=1}^{K} |x(k)|^2 \leq M^2$$

for all K.

Hence $\sum_{k=1}^{\infty} |x(k)|^2$ converges (the partial sums are bounded above and the terms are non negative).

So $x \in \ell_2$.

Does $x_n \to x$?

Let $\varepsilon > 0$. Pick N so that if $n, m \geq N$,

$\|x_n - x_m\|_2 < \varepsilon$. Suppose $n \geq N$.

For each K

$$\sum_{k=1}^{K} |x(k) - x_n(k)|^2 = \lim_{m \to \infty} \sum_{k=1}^{K} |x_m(k) - x_n(k)|^2.$$

$$\leq \limsup_{m \to \infty} \|x_m - x_n\|_2^2$$

Since $\|x_m - x_n\|_2 < \varepsilon$ if $m \geq N$, it follows that $\limsup_{m\to\infty} \|x_m - x_n\|_2 \le \varepsilon$.

Hence for each K,

$$\sum_{k=1}^{K} |x(k) - x_n(k)|^2 \le \varepsilon^2.$$

Consequently, $\|x - x_n\|_2^2 = \sum_{k=1}^{\infty} |x(k) - x_n(k)|^2 \le \varepsilon^2$.

So: if $n \geq N$ then $\|x - x_n\|_2 \le \varepsilon$.

Hence $x_n \to x$ in l_2.