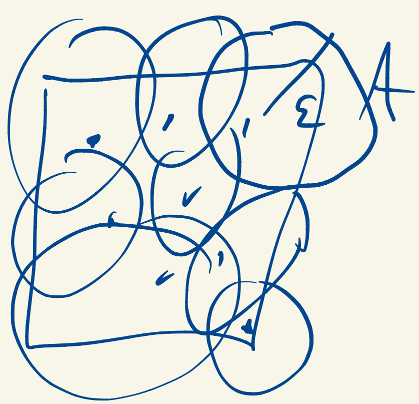
Def: A set $A \subseteq X$ is, like, <u>totally bounded</u> if for every $\varepsilon > 0$ there are finitely many points $x_1, \dots, x_n \in X$ such that

$$A \subseteq \bigcup_{k=1}^{n} B_\varepsilon(x_k).$$

Such a collection of points is called an <u>ε-net</u>.



$\varepsilon > 0$

total boundedness $\Rightarrow$ boundedness

1-net

bounded $\Rightarrow$ totally bounded ? No!

$\ell_1 \quad A = \{e_k\} \quad e_k = (0, \dots, 0, 1, 0, \dots)$

k^{th}

Claim A does not admit

a 1 net.

If $\underline{j \neq k}$ $\quad \|e_j - e_k\|_1 = 2$

So any $B_1(x) \subseteq \ell_1$ can contain at most

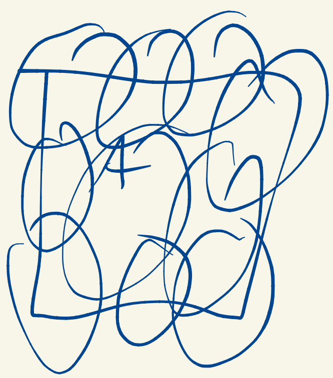
one e_j. (if $y, z \in B_1(x)$ $d(y,z) < 2$)

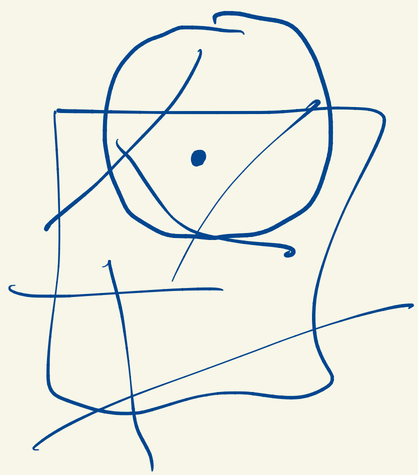
So any finite collection of 1-balls contains at most finitely many e_k's. So there is no 1-net.

The set A is bounded (it's contained in a ball of radius 10 centered at 0) but not totally bounded.

Lemma: A set $A \subseteq X$ is totally bounded iff for every $\varepsilon > 0$ there exist $A_1, \ldots, A_n$ with $\operatorname{diam} A_k < \varepsilon$ and $A \subseteq \bigcup_{k=1}^{n} A_k$.

Pf:





See text.

Cor: $[0,1]$ is totally bounded.

Use subintervals $\underbrace{\left[\frac{k-1}{n}, \frac{k}{n}\right]}_{I_{k,n}}$ $\quad 1 \le k \le n$

$\operatorname{diam}(I_{k,n}) = \frac{1}{n}$

$$\bigcup_{k=1}^{n} I_{k,n} = [0,1]$$

Exercise: $[-R, R]$ is totally bounded for all $R > 0$.

Exercise: If $B \subseteq A$ and A is totally bounded then B is totally bounded.

Exercise: bounded subsets of $\mathbb{R}$ are totally bounded

Total boundedness is closely connected to Cauchy sequences.

Lemma: Suppose (x_n) is Cauchy. Then $\{x_n : n \in \mathbb{N}\}$ is totally bounded.

Pf: Let $\varepsilon > 0$. [Job: find an ε net]. There exists N such that if $n, m \geq N$ then $d(x_n, x_m) < \varepsilon$.

I claim that $\{x_1, x_2, \ldots, x_N\}$ is an ε net.

Indeed if $n \geq N$ then $d(x_N, x_n) < \varepsilon$ and $x_n \in B_\varepsilon(x_N)$. Otherwise $x_n \in B_\varepsilon(x_n)$.

Lemma: Given a sequence (x_n), if $\{x_n : n \in \mathbb{N}\}$ is totally bounded, then the sequence admits a Cauchy subsequence.

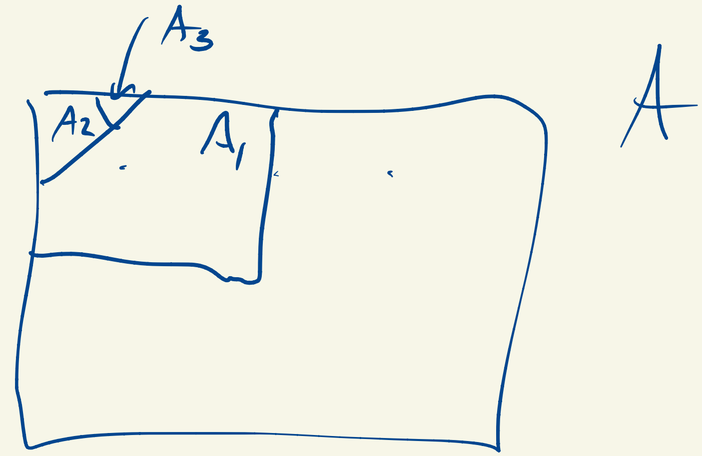
Pf: If $A = \{x_n : n \in \mathbb{N}\}$ is finite then we can extract a constant and hence Cauchy subsequence. Otherwise, suppose A is infinite. Since A is totally bounded there is a subset A_1 with $\operatorname{diam} A_1 < 1$ and such that A_1 contains infinitely many points of A.

Since A_1 is totally bounded and infinite, it admits an infinite subset A_2 with $\operatorname{diam} A_2 < \frac{1}{2}$.

Continuing inductively we can find nested sets

$$A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$$

with $\operatorname{diam} A_k < \frac{1}{k}$.



Pick n_1 with $x_{n_1} \in A_1$.

(x_n)

Pick $n_2 > n_1$ with $x_{n_2} \in A_2$.

This is possible since A_2 is infinite.

$\underbrace{x_1, x_2, \dots, x_{n_1}}$

Continuing inductively we select indices $n_1 < n_2 < n_3 < \cdots$

with $x_{n_k} \in A_k$.

I claim (x_{n_k}) is Cauchy. Indeed let $\varepsilon > 0$.

Pick K so that $\frac{1}{K} < \varepsilon$.

Then if $k, \ell \geq K$ then $x_k \in A_k \subseteq A_K$

$x_\ell \in A_\ell \subseteq A_K$

and hence $d(x_k, x_\ell) < \frac{1}{K} < \varepsilon$.

Thm: A set $A \subseteq X$ is totally bounded iff every sequence in A admits a Cauchy subsequence.

Pf: Suppose A is totally bounded. Then if (x_n) is a sequence in A, $\{x_n : n \in \mathbb{N}\} \subseteq A$ is totally bounded and the previous result implies the sequence has a Cauchy subsequence.

Suppose A is not totally bounded.

[Job: Find a sequence with no Cauchy subsequence]

Since A is not totally bounded there is $\varepsilon > 0$ such that A does not admit an ε-net.

Let $a_1 \in A$. I claim that $A \setminus B_\varepsilon(a_1) \neq \emptyset$.

This is true, for otherwise $\{a_1\}$ is an ε-net.

Pick $a_2 \in A \setminus B_\varepsilon(a_1)$. Since $\{a_1, a_2\}$ is not an ε-net, $A \setminus \bigcup_{k=1}^{2} B_\varepsilon(a_k) \neq \emptyset$.

Continuing inductively we can find a sequence (a_k) with each a_k in A and $d(a_k, a_\ell) \geq \varepsilon$ if $k \neq \ell$.

No subsequence can be Cauchy, for any Cauchy subsequence would contain two terms at distance no more than $\varepsilon/2$ from each other.