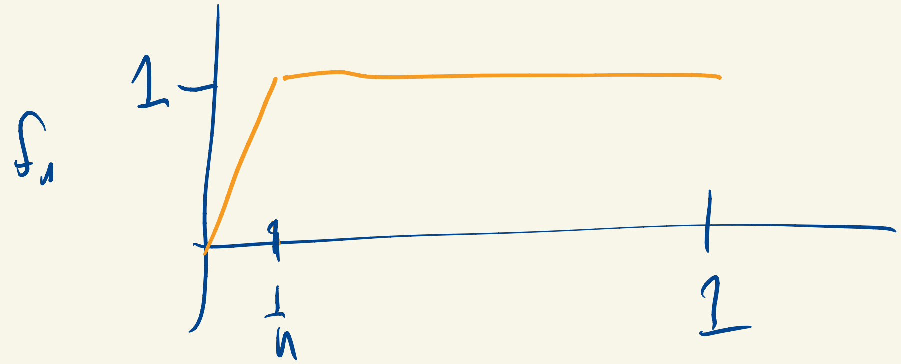
$p = 1$?



$f_n \xrightarrow{L_1} 1$

$F(f_n) = 0$

$F(1) = 1$

$F(f_n) \to F(1)$

No: not continuous.

$f_n \to f$

$F(f_n) \not\to F(f)$

$G(f) = \int_0^1 f(x)\,dx$

$G : C[0,1] \to \mathbb{R}$

Is G continuous if $C[0,1]$ has the L_1 norm?

Given $f, g \in C[0,1]$

$$d_{\mathbb{R}}(G(f), G(g))$$

$$\begin{aligned} |G(f) - G(g)| &= \left| \int_0^1 f(x)\,dx - \int_0^1 g(x)\,dx \right| \\ &= \left| \int_0^1 (f(x) - g(x))\,dx \right| \\ &\leq \int_0^1 |f(x) - g(x)|\,dx \\ &= \|f-g\|_1 \\ &= d_1(f,g) \end{aligned}$$

Let $\varepsilon > 0$. Pick $\delta = \varepsilon$. Then if $d_1(f,g) < \delta$

the inequality above shows $d_{\mathbb{R}}(G(f), G(g)) \le \|f-g\|_1 < \delta = \varepsilon$.

Alt: If $f_n \xrightarrow{L_1} f$ then $|G(f_n) - G(f)| \le \|f_n - f\|_1 \to 0$,

So $G(f_n) \to G(f)$.

Exercise: Show

$$(C[0,1], L_\infty) \longrightarrow (C[0,1], L_1)$$
$$f \longmapsto f$$

is continuous. $f_n \xrightarrow{L_\infty} f \Rightarrow f_n \xrightarrow{L_1} f$

$G : (C[0,1], L_\infty) \to \mathbb{R}$ is also continuous

$$C[0,1], L_\infty \longrightarrow C[0,1], L_1 \longrightarrow \mathbb{R}$$

Exercise: If $f: X \to Y$ and $g: Y \to Z$ are continuous then

$g \circ f : X \to Z$ is also continuous.

(Two ways! ε-δ, sequence)

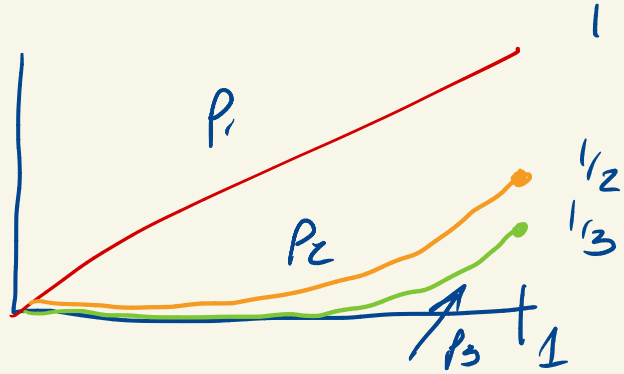
E.g. $P[0,1] \subseteq C[0,1]$ (L_∞)

$D: P[0,1] \to P[0,1]$

$D(p) = p'$ ⟸ derivative of p.

$p_n(x) = \frac{1}{n} x^n$

$|p_n(x)| \leq \frac{1}{n} \quad \forall n$

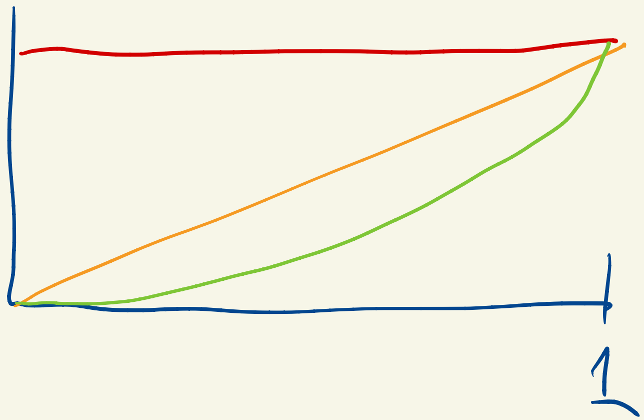


$\|p_n\|_\infty \leq \frac{1}{n}$

$\|p_n - 0\|_\infty \leq \frac{1}{n}$

$p_n \xrightarrow{L_\infty} 0$

$$D(p_n) = x^{n-1}$$

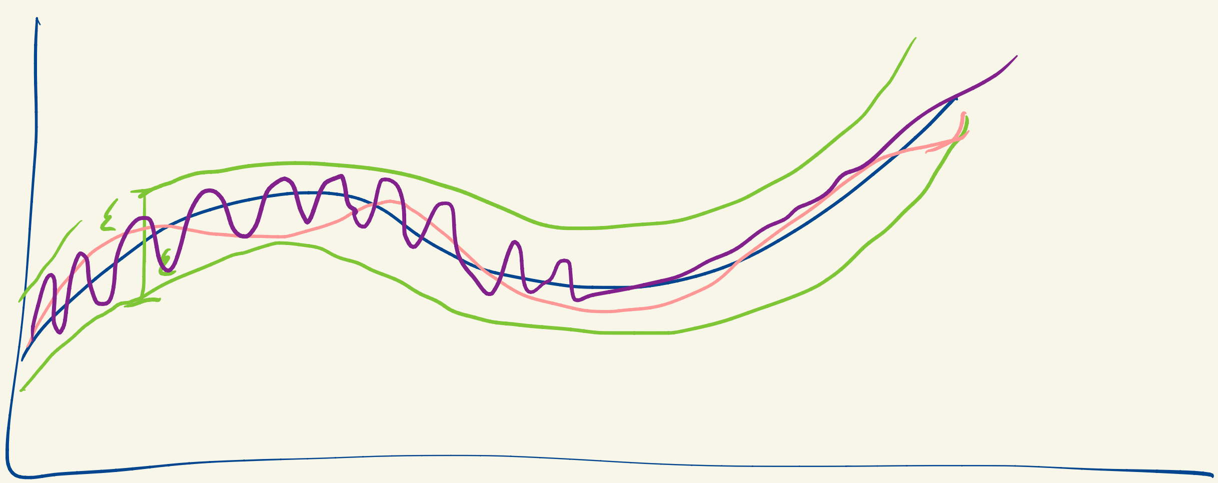


$$D(p_n) \xrightarrow[L_\infty]{?} D(0)$$

$$D(p_n)(1) = 1$$

$$f_n \xrightarrow{L_\infty} g \Rightarrow f_n(x) \to g(x)$$

$D(p_n) \not\to D(0)$ so D is not continuous.



Suppose $f: X \to Y$ is continuous and a bijection.

Must f^{-1} be continuous? No!

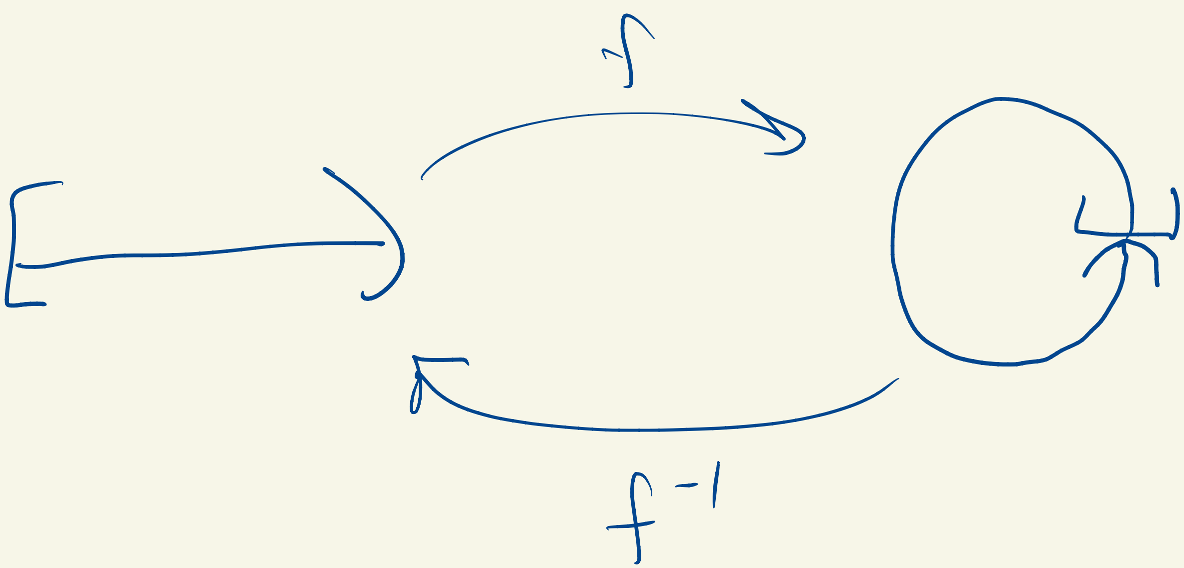
"homeomorphism"

E.g.

$$[0, 2\pi) \xrightarrow{f} S^1$$

$S^1 \to$ unit circle in $\mathbb{R}^2$

$$f(\theta) = (\cos\theta, \sin\theta)$$



Claim: f^{-1} is not continuous.

I'll show $\exists$ x_n's in S^1

$$x_n \to x$$

$$f^{-1}(x_n) \not\to f^{-1}(x)$$

$$x_n = \left(\cos\left(-\frac{1}{n}\right), \sin\left(-\frac{1}{n}\right)\right) \in S^1$$

$$f^{-1}(x_n) = 2\pi - \frac{1}{n}$$

$$x_n \to (1,0) \qquad\qquad x_n \to (1,0)$$

$$f^{-1}((1,0)) = 0$$

$$f^{-1}(x_n) \to 2\pi \neq 0 = f^{-1}((1,0))$$

So f^{-1} is not continuous.

Def: A function $f: X \to Y$ is an isometry if

for all $x_1, x_2 \in X$ $\quad d(x_1, x_2) = d(f(x_1), f(x_2))$

iso $\Rightarrow$ same

metry $\Rightarrow$ distance

e.g $f: \mathbb{R} \to \mathbb{R}$

$f(x) = x$

$f(x) = x+1$

$f(x) = -x$

$f(x) = -x+18$

Exercise: Show that an isometry $f: \mathbb{R} \to \mathbb{R}$ is uniquely determined by its action on two points.

That is if $x_1, x_2 \in \mathbb{R}$ $x_1 \neq x_2$ and if $f_1, f_2: \mathbb{R} \to \mathbb{R}$ are isometries with $f_1(x_i) = f_2(x_i)$ $i = 1, 2$

then $f_1 = f_2$. Use this to show that every isometry $f: \mathbb{R} \to \mathbb{R}$ has the form

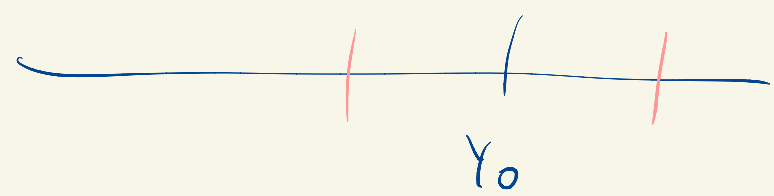
$$f(x) = x + c \quad \text{or} \quad f(x) = -x + c \quad \text{for}$$

some $c \in \mathbb{R}$.

$f: \mathbb{R} \to \mathbb{R}$ isometry

$$f(0) = y_0$$

$$f(1) = \begin{cases} y_0 + 1 \\ y_0 - 1 \end{cases}$$



$$f(x) = x + y_0$$

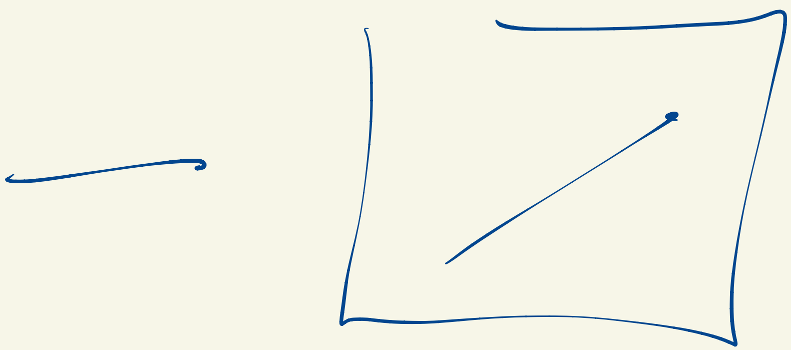
Are isometries always injective? Yes! If $x_1 \neq x_2$ then

$$d(x_1, x_2) > 0$$

so $d(f(x_1), f(x_2)) = d(x_1, x_2) > 0$

So $f(x_1) \neq f(x_2)$.

Surjective? No! Put a line in the plane



Exercise: A surjective isometry always has a continuous inverse

(which is an isometry)

Note: isometries are continuous for if $x_n \to x$

then $d(f(x_n), f(x)) = d(x_n, x) \to 0$ so

$f(x_n) \to f(x)$.

linear spaces $\longleftrightarrow$ linear isomorphisms
(bijective linear maps)

topology $\longleftrightarrow$ homeomorphism

groups $\longleftrightarrow$ group isomorphism

$\mathbb{Z}/2\mathbb{Z}, +$

$\{1, -1\}, \cdot$

Key property of $\mathbb{R}$: every bounded sequence has a convergent subsequence.

Is this true for metric spaces? No. Two things can go wrong

1) Completeness $\underbrace{3,\ 3.1,\ 3.14,\ \ldots}_{}$

bounded sequence in $\underline{\underline{\mathbb{Q}}}$ not converge

2) Something else. (Total boundedness)

$e_n \in \ell_\infty$

$e_n = (0, 0, \ldots, 1, 0, \ldots)$

$\uparrow$ n'th position

$\{e_n\}$ is bounded in ℓ_∞

Do the e_n's converge?

$\| e_n - e_m \|_\infty = 1$

$n \neq m$