Closed sets are completely determined by convergence of sequences.

Two metrics might be different but still determine the same convergent sequences. In this case, they determine the same open and closed sets.

Exercise: An arbitrary union of open sets is open.

Exercise: An arbitrary intersection of closed sets is closed.

An arbitrary intersection of open sets need not be open. $(-\frac{1}{n}, \frac{1}{n})$

Def: Given a set A in a metric space, $\bar{A}$ (the closure of A) is the intersection of all closed sets containing A.

Observe: $\overline{A}$ is closed! It is the smallest closed set containing A,

Prop: Let $A \subseteq X$ and let $x \in X$. TFAE Yes!

1) $x \in \overline{A}$
2) $\forall \varepsilon > 0 \quad B_\varepsilon(x) \cap A \neq \phi$ (i.e. $\exists y \in A$ with $d(y,x) < \varepsilon$)
3) there is a sequence in A converging to x.

Pf: 1) $\Rightarrow$ 2) via !2) $\Rightarrow$!1)

Suppose for some $\varepsilon > 0$ $B_\varepsilon(x) \cap A = \phi$. Then $B_\varepsilon(x)^c \supseteq A$.

Since $B_\varepsilon(x)^c$ is closed and contains A, $\overline{A} \subseteq B_\varepsilon(x)^c$.

Hence $x \notin \overline{A}$.

2) $\Rightarrow$ 3) For each $n \in \mathbb{N}$ pick $x_n \in B_{1/n}(x) \cap A$.

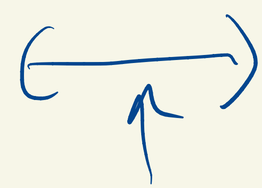
Then (x_n) is a sequence in A with $d(x,x_n) < \frac{1}{n} \to 0$.

So $x_n \to x$.

3) $\Rightarrow$ 1)

Suppose (x_n) is a sequence in A converging to x.
Then it is also a sequence in $\bar{A}$. Hence the limit of the sequence (x!) is also in $\bar{A}$.

Note: $\overline{\mathbb{Q}} = \mathbb{R}$.



$\bar{A}$ is the set of points in X that can be approximated arbitrarily well by things in A.

Def: We say A is dense in X if $\overline{A} = X$.

A space X is separable if it admits a countable dense subset.

Countable is manageable, separable is almost as good.

$P[0,1] \subseteq C[0,1]$

$\uparrow$

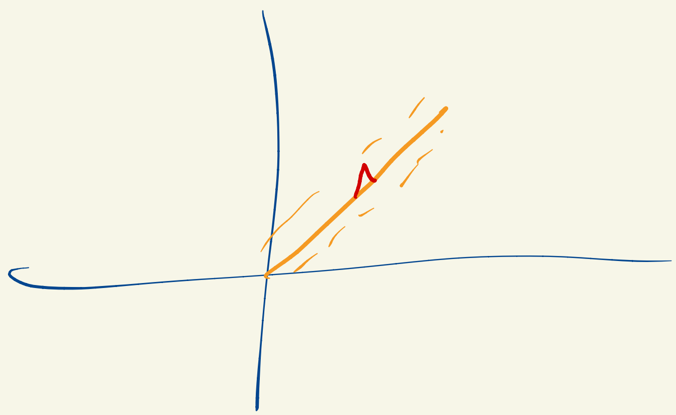
polynomials

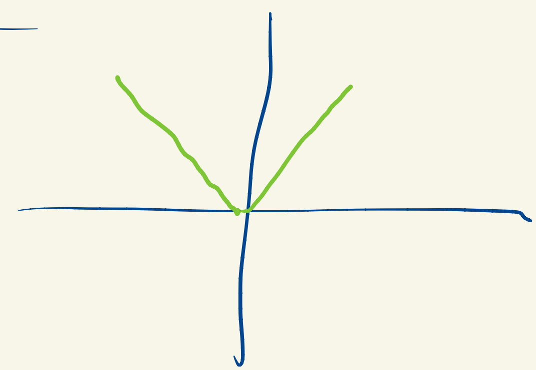
Is $P[0,1]$ open?

closed?

dense?

We'll prove this





Indeed, polynomials with rational coefficients are dense in $C[0,1]$. (Which leads to $C[0,1]$ being separable)

Metrics on related spaces

If $A \subseteq X$ then A is a metric space in its own right.

$$d_A(x,y) = d_X(x,y)$$

Exercise: $U \subseteq A$ is open $\iff \exists\, V \subseteq X$ that is open and $U = A \cap V$.

$W \subseteq A$ is closed $\iff \exists\, Z \subseteq X$ that is closed and $W = A \cap Z$.

(x_n) a seq. in A as in X

$$x_n \xrightarrow{A} x \iff x_n \xrightarrow{X} x$$

Product spaces

X, Y metric spaces d_X, d_Y

$$X \times Y = \{ (x,y) : x \in X, y \in Y \}$$

$$(x_n, y_n) \longrightarrow (x, y) \iff x_n \to x, \; y_n \to y$$

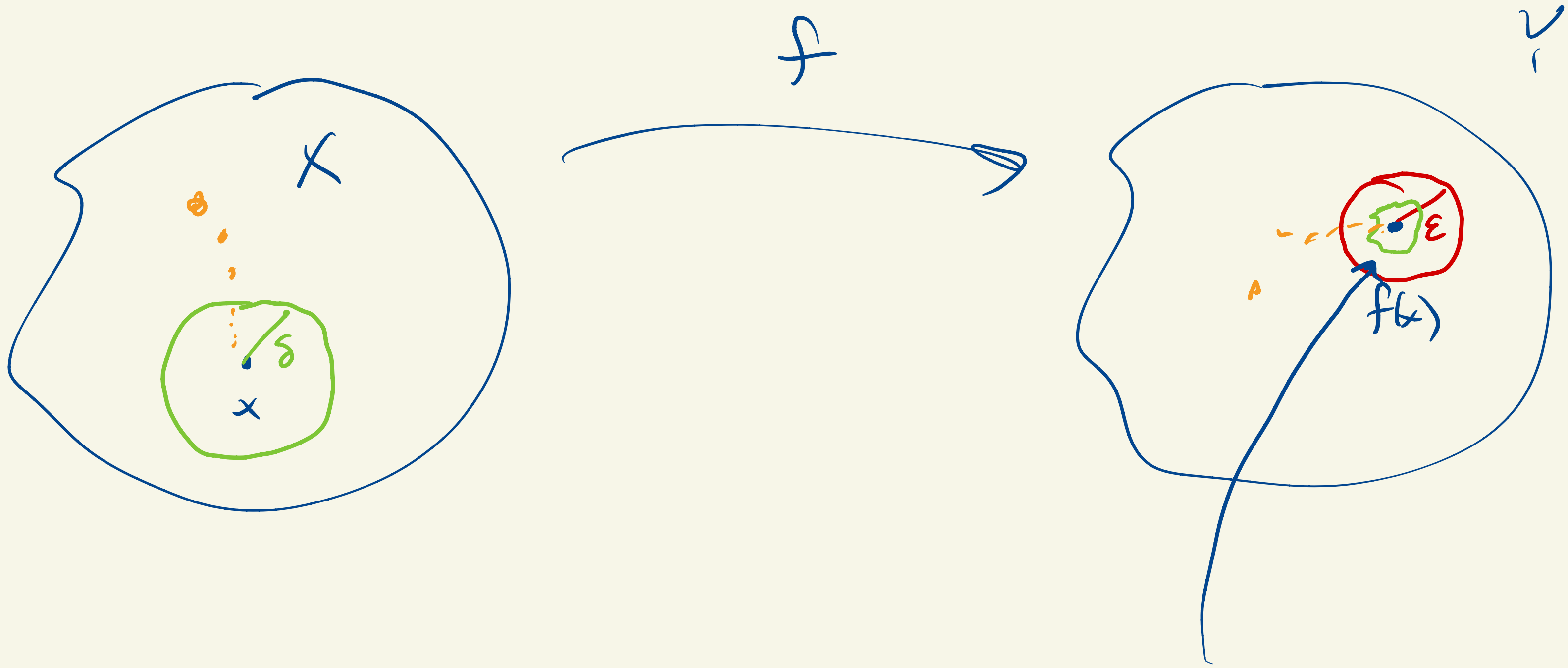
$$d_{X \times Y}\big((x_0, y_0), (x_1, y_1) \big) = \boxed{\begin{cases} d_X(x_0, x_1) + d_Y(y_0, y_1) \\ \sqrt{d_X^2(x_0, x_1) + d_Y^2(y_0, y_1)} \\ \max\big(d_X(x_0, x_1), d_Y(y_0, y_1) \big) \end{cases}}$$

all metrics

Continuity

Def: We say $f: X \to Y$ is continuous at $x \in X$ if for all $\varepsilon > 0$ there is $\delta > 0$ such that for all $\underbrace{y \in B_\delta(x)}_{y \text{ with } d_X(x,y) < \delta}$, $d_Y(f(x), f(y)) < \varepsilon$.

$f: \mathbb{R} \to \mathbb{R}$ is cts at x if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all y with $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$.



A function f is continuous if it is continuous at every point.

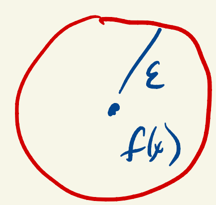
Def: A function $f: X \to Y$ is sequentially continuous at $x \in X$ if whenever $x_n \to x$, $f(x_n) \to f(x)$.

Prop: f is continuous at x $\Longleftrightarrow$ f is sequentially continuous at x.

Pf: Suppose f is continuous and $x_n \to x$.

Let $\varepsilon > 0$. Then since f is cts, there exists $\delta > 0$ such that $f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$. Since $x_n \to x$ there exists N such that if $n \geq N$, $x_n \in B_\delta(x)$. But then if $n \geq N$, $f(x_n) \in f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$. Hence $f(x_n) \to f(x)$.

Conversely, suppose f is not continuous at x. Then there exists $\varepsilon > 0$ such that for all $\delta > 0$ $f(B_\delta(x)) \not\subseteq B_\varepsilon(x)$. So for each $n \in \mathbb{N}$ we can pick $x_n \in B_{1/n}(x)$ such that $f(x_n) \notin B_\varepsilon(f(x))$. Observe $x_n \to x$. But $f(x_n) \to f(x)$ since $B_\varepsilon(f(x))$ contains no terms of the sequence.



E.g. $F : C[0,1] \to \mathbb{R}$

$F(f) = f(0)$

$F(\exp) = 1 \qquad (e^0 = 1)$

$F(\sin) = 0$

Is F cts if $C[0,1]$ is given the L_p norm?

$p = \infty$? yes! Suppose $f_n \xrightarrow{L_\infty} f$.

Then $\|f - f_n\|_\infty \to 0$

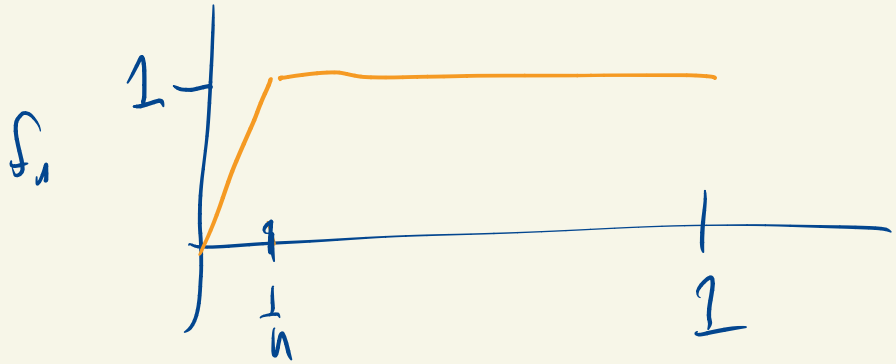
But $|f_n(0) - f(0)| \leq \|f_n - f\|$.

So $|f_n(0) - f(0)| \to 0$.

So $f_n(0) \to f(0)$.

So $F(f_n) \to F(f)$.

$p = 1$?



$f_n \xrightarrow{L_1} 1$

$F(f_n) = 0$

$F(1) = 1$

$F(f_n) \to F(1)$

No: not continuous.