Def: Let (x_n) be a sequence in a metric space X.

We say $x_n \to x$ ((x_n) converges to x) if

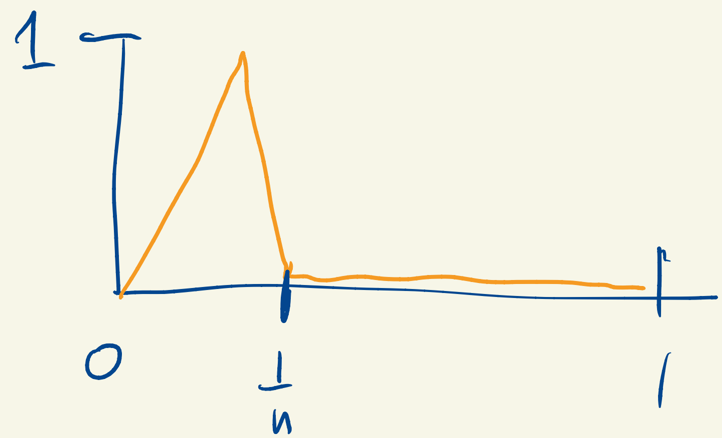
for all $\varepsilon > 0$ $\exists N$ so if $n \geq N$ $d(x_n, x) < \varepsilon$,

Def: A sequence is Cauchy if $\forall \varepsilon > 0$ $\exists N$ s.t.

$\forall n, m \geq N$ $d(x_n, x_m) < \varepsilon$.

Consider f_n

$C[0,1]$



Does $f_n \to 0$? Answer depends on norm

$\|f_n\|_\infty = 1$ for all n. $d(f_n, 0) = 1$

$\|f_n - 0\|_\infty$

Exercise $x_n \to x \iff d(x_n, x) \to 0$

$f_n \not\to 0$

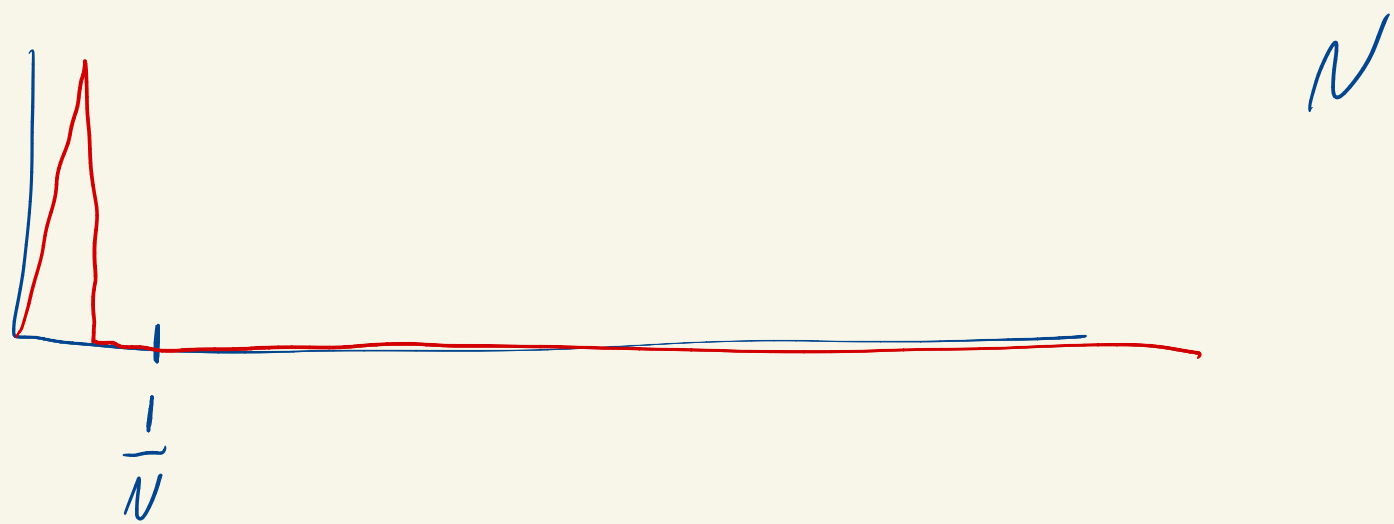
$\|f_n\|_1 = \frac{1}{2n}$ $\qquad d_1(f_n, 0) = \frac{1}{2n} \to 0$

L_p

$f_n \xrightarrow{L_1} 0$

Exercise: Determine if $f_n \to 0$ in L_p $1 < p < \infty$

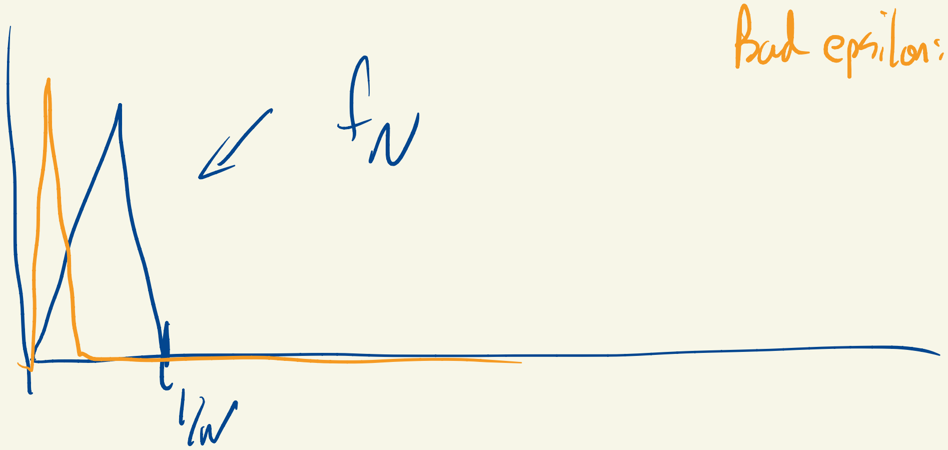
Exercise: Show that convergent sequences are Cauchy.



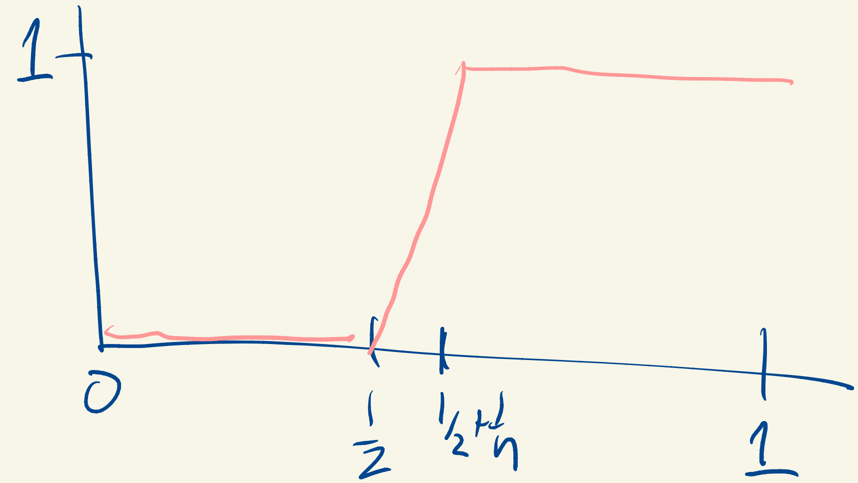
Exercise: Show (f_n) is not convergent in L_∞ sense.

(Hint: show it is not Cauchy.)

Bad epsilon: $1/2$



Consider g_n



$C[0,1]$

This sequence is Cauchy in L_1.

Given some N, if $n, m \geq N$ then $(g_n - g_m)(x) = 0$ if $x \leq \frac{1}{2}$

or if $x \geq \frac{1}{2} + \frac{1}{N}$

$$|g_n(x)| \leq 1$$

$$\| g_n - g_m \|_1 = \int_0^1 |(g_n - g_m)(x)| \, dx$$

$$= \int_{1/2}^{1/2+\frac{1}{N}} |(g_1 - g_m)(x)| \, dx$$

$$\leq \int_{1/2}^{1/2+\frac{1}{N}} 2 \, dx$$

$$= \frac{2}{N}$$

Is the sequence convergent in $C[0,1]$?

The sequence does not have a limit in $C[0,1]$.

Suppose not and let g be the limit.

Pick $x_0 > \frac{1}{2}$. Then $g_n = 1$ on $[x_0, 1]$ for n sufficiently large

$$\left(\frac{1}{2} + \frac{1}{n} < x_0 \right)$$

$$\int_{x_0}^{1} |g(x)-1|\,dx = \int_{x_0}^{1} |g(x)-g_n(x)|\,dx \leq \|g-g_n\|_1 \longrightarrow 0$$

(n large enough)

So $\int_{x_0}^{1} |g(x)-1|\,dx = 0$. Hence $|g(x)-1| = 0$

for all $x \in [x_0, 1]$.

If f is continuous on $[a,b]$

and $f \geq 0$ and

$\int_a^b f(x)\,dx = 0$ then $f = 0$.

If g exists then $g(x) = 1$ for all $x > \frac{1}{2}$

Same argument: $g(x)=0$ for all $x<\frac{1}{2}$.

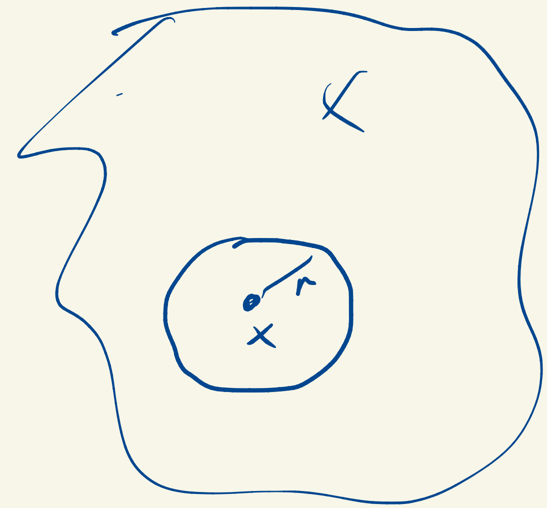
There is no such $g\in C[0,1]$

Def: Let X be a metric space

Given $x\in X$ and $r>0$

$$B_r(x)=\{y\in X: d(x,y)<r\}$$

↳ ball of radius r centered at x.



Similarly

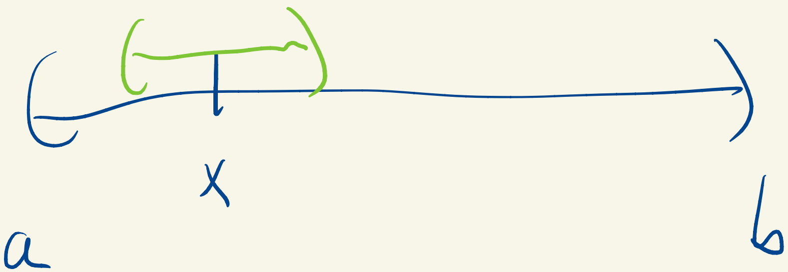
$$\overline{B_r}(x)=\{y\in X: d(x,y)\leq r\}$$

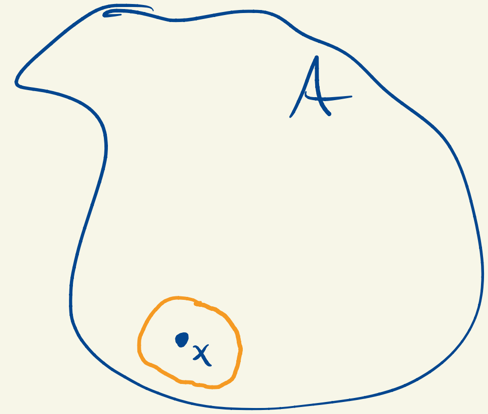
Def: A set $A\subseteq X$ is <u>open</u> if for all $x\in A$

there exists $r>0$ s.t. $B_r(x)\subseteq A$

Examples:

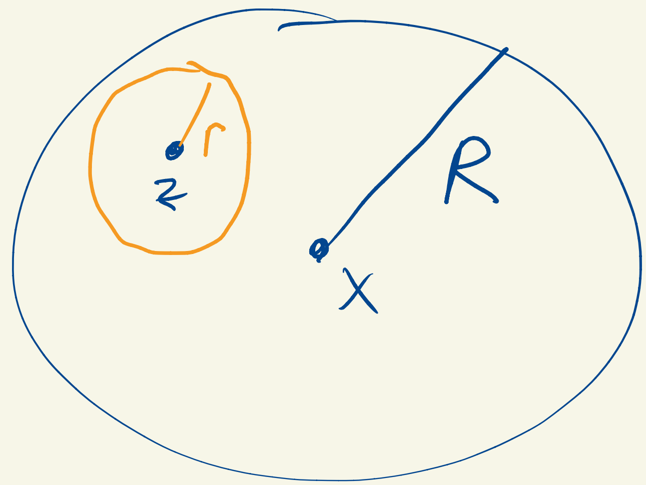
$(a,b) \subseteq \mathbb{R}$





$\emptyset \subseteq X$

$B_R(x) \subseteq X$



$r = R - d(z,x)$

(use Δ ineq.

$A = \{ f \in C[0,1] : f(0) > 0 \}$

Is A open in $C[0,1]$?

Yes in L_∞ sense.

Given $f \in A$ let $r = f(0) > 0$.

Exercise: show $B_r(f) \subseteq A$.

If $g \in B_r(f)$ then $\|f-g\|_\infty < r$

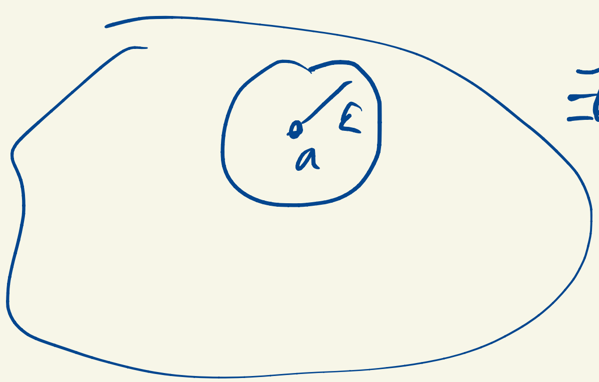
so $f(0) - g(0) < r$

so $g(0) > f(0) - r$
$\quad\quad = 0$

But this is false in the L_1 norm,

Note: If $A \subseteq X$ and if $x \in A$ and (x_n) is a sequence in A^c with $x_n \to x$, then A is not open.

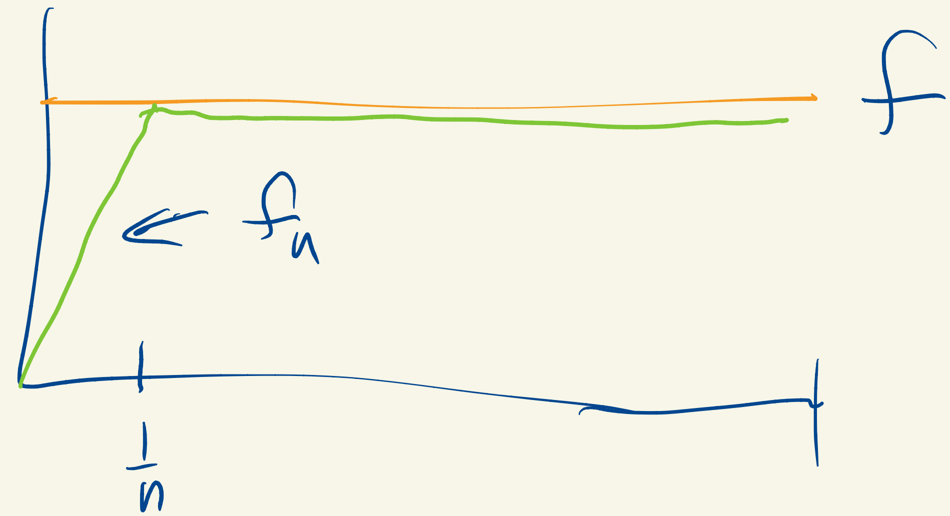
Indeed, if $a \in A$ and A is open and $a_n \to a$ then there is a tail of the sequence contained in A.



$\exists N$, $a_n \in B_\varepsilon(a)$ for $n \geq N_0$

$f = 1 \in A$

$C[0,1]$



Build a sequence in A^c converging to f.

Each $f_n \in A^c$. $f_n \to f$ since $d(f_n, f) = \frac{1}{2}\frac{1}{n} \to 0$

Lemma: Suppose $A \subseteq X$ is not open. Then there is a sequence in A^c converging to some $x \in A$.

Pf: Since A is not open there exist $x \in A$ such that for all $\varepsilon > 0$, $B_\varepsilon(x)$ is not contained in A.

Hence for each $n \in \mathbb{N}$ we can select $x_n \in A^c \cap B_{\frac{1}{n}}(x)$.

For each n, $d(x, x_n) < \frac{1}{n} \to 0$.

So $x_n \to x$.

Def: A set A is <u>closed</u> if A^c is open.

$[0,1]$ is closed

$[0,1]^c = (-\infty, 0) \cup (1, \infty)$

Prop: A set A is closed iff whenever (x_n) is a sequence in A converging to some limit x, in fact $x \in A$.

Pf: Suppose A is closed and $y \notin A$. Then there exists $r > 0$ with $B_r(y) \subseteq A^c$. Hence any sequence in X converging to y must contain terms in $B_r(y)$ and in particular terms in A^c. So no sequence in A can converge to y.

Conversely, suppose A is not closed, so A^c is not open. Then there exists $x \in A^c$ and a sequence in $(A^c)^c$ converging to x. That is, there is a sequence in A converging to a point $x \notin A$.