Norms on $C[0,1]$

$$\|f\|_1 = \int_0^1 |f(x)|\,dx \qquad \left(\sim \sum_{k=1}^{\infty} |x_k|\right)$$

$$\|f\|_2 = \left[\int_0^1 |f(x)|^2\,dx\right]^{1/2}$$

$$\|f\|_p = \left[\int_0^1 |f(x)|^p\,dx\right]^{1/p} \qquad 1 \le p < \infty$$

$$\|f\|_\infty = \sup_{x\in[0,1]} |f(x)| = \max_{x\in[0,1]} |f(x)|$$

default

Triangle Inequalities!

arithmetic mean $\quad$ geometric mean

Prop (special case of AM–GM inequality)

For all $a, b \in \mathbb{R}$

$$|a|\,|b| \leq \frac{1}{2}\left(a^2 + b^2\right)$$

Pf: It suffice to suppose $a, b \geq 0$.

Since $(a-b)^2 \geq 0$ we have

$$a^2 - 2ab + b^2 \geq 0$$

and hence

$$ab \leq \frac{1}{2}\left(a^2 + b^2\right). \qquad \square$$

$a^2 = \alpha \qquad a = \alpha^{1/2}$

$\beta^2 = \beta \qquad b = \beta^{1/2}$

$$\alpha^{1/2}\beta^{1/2} \leq \frac{1}{2}(\alpha + \beta)$$

$$(\alpha\beta)^{1/2} \leq \frac{1}{2}(\alpha + \beta)$$

Prop: (Cauchy Shwartz Inequality) $\left(\vec{x} \cdot \vec{y} \leq \|\vec{x}\| \, \|\vec{y}\| \right)$

For all $x, y \in \mathbb{R}^n$

$$\sum_{k=1}^{n} |x_k y_k| \leq \|x\|_2 \|y\|_2$$

Pf: Suppose first that $\|x\|_2 = \|y\|_2 = 1$.

Then

$$\sum_{k=1}^{n} |x_k y_k| \leq \sum_{k=1}^{n} \frac{1}{2}\left(x_k^2 + y_k^2\right) = \frac{1}{2}\left[\sum_{k=1}^{n} x_k^2 + \sum_{k=1}^{n} y_k^2\right]$$

$$= \frac{1}{2}\left[\|x\|_2^2 + \|y\|_2^2\right]$$

$$= 1 = \|x\|_2 \|y\|_2$$

If $x = 0$ or $y = 0$ then the inequality is trivial.

Otherwise, let $z = \dfrac{x}{\|x\|_2}$ and let $w = \dfrac{y}{\|y\|_2}$, so $\|z\|_2 = \|w\|_2 = 1$.

We just saw that

$$\sum_{k=1}^{n} |z_k w_k| \leq 1$$

and hence $$\sum_{k=1}^{n} \frac{|x_k y_k|}{\|x\|_2 \|y\|_2} \leq 1.$$

$\square$

Prop (CS ineq for ℓ_2)

$\forall\, x, y \in \ell_2$ $\qquad \displaystyle\sum_{k=1}^{\infty} |x_k y_k| \leq \|x\|_2 \|y\|_2.$

Pf: For each $N \in \mathbb{N}$

$$\sum_{k=1}^{N} |x_k y_k| \leq \left(\sum_{k=1}^{N} |x_k|^2\right)^{1/2} \left(\sum_{k=1}^{N} |y_k|^2\right)^{1/2}$$

$$\leq \|x\|_2 \|y\|_2.$$

Now take a limit as $N \to \infty$.

Cor: For all $x, y \in \ell_2$, $x+y \in \ell_2$ and

$$\|x+y\|_2 \leq \|x\|_2 + \|y\|_2 .$$

Pf: For each k,

$$(x_k + y_k)^2 = x_k^2 + 2x_k y_k + y_k^2 .$$

Hence

$$\sum_{k=1}^{\infty} (x_k + y_k)^2 = \|x\|_2^2 + 2\sum_{k=1}^{\infty} x_k y_k + \|y\|_2^2$$

$$\leq \|x\|_2^2 + 2\|x\|_2 \|y\|_2 + \|y\|_2^2$$

$$= \left(\|x\|_2 + \|y\|_2\right)^2 .$$

Now take square roots.

How about l_p?

l_1 $$\sum_{k=1}^{\infty} |x_k + y_k| \leq \sum_{k=1}^{\infty} |x_k| + \sum_{k=1}^{\infty} |y_k|$$

l_∞ $$\sup_k |x_k + y_k| \leq \sup_k |x_k| + \sup_k |y_k|$$

$$|x_k + y_k| \leq |x_k| + |y_k|$$

$$\leq \sup_n |x_n| + \sup_n |y_n|$$

$$\Rightarrow \quad \sup |x_k + y_k| \leq$$

What about the other values of p?

Observation: given $x, y \in \ell_p$, then $x+y \in \ell_p$

$$|x_k + y_k| \leq |x_k| + |y_k| \leq 2\max(|x_k|, |y_k|)$$

$$|x_k+y_k|^p \leq 2^p \max(|x_k|^p, |y_k|^p)$$
$$\leq 2^p \left(|x_k|^p + |y_k|^p \right)$$

Hence

$$\|x+y\|_p^p \leq 2^p \left(\|x\|_p^p + \|y\|_p^p \right)$$

There is a variation of the Cauchy-Schwartz inequality that holds for ℓ_p.

$$\sum_{k=1}^{\infty} |x_k y_k| \leq \|x\|_2 \|y\|_2$$

Hölder's Inequality.

p, q

$$\underbrace{\frac{1}{p} + \frac{1}{q} = 1}_{\text{Hölder conjugate exponents}}$$

Thm: Hölder's Inequality:

Suppose $1 < p < \infty$ and q satisfies $\frac{1}{p} + \frac{1}{q} = 1$

If $x \in \ell_p$ and $y \in \ell_q$ then

$$\sum_{k=1}^{\infty} |x_k y_k| \leq \|x\|_p \|y\|_q$$

The ℓ_p triangle inequality follows from Hölder's Inequality

Thm (Triangle Inequality for ℓ_p) For all $x, y \in \ell_p$, $1 < p < \infty$

$$\|x+y\|_p \leq \|x\|_p + \|y\|_p .$$

Pf: Observe

$$\|x+y\|_p^p = \sum_{k=1}^{\infty} |x_k + y_k|^p = \sum_{k=1}^{\infty} |x_k + y_k| \, |x_k + y_k|^{p-1}$$

$$\leq \sum_{k=1}^{\infty} |x_k| \, |x_k + y_k|^{p-1} + \sum_{k=1}^{\infty} |y_k| \, |x_k + y_k|^{p-1}$$

$$\leq \|x\|_p \, \|(|x_k + y_k|^{p-1})\|_q + \|y\|_p \, \|(|x_k + y_k|^{p-1})\|_q$$

with $\frac{1}{p} + \frac{1}{q} = 1$. Note: $(p-1)q = p$ and hence

$$\|(|x_k + y_k|)^{p-1}\|_q^q = \sum_{k=1}^{\infty} |x_k + y_k|^{(p-1)q} = \sum_{k=1}^{\infty} |x_k + y_k|^p = \|x+y\|_p^p$$

Here

$$\|x+y\|_p^p \leq \left(\|x\|_p + \|y\|_p\right) \|x+y\|_p^{p/q}$$

If $\|x+y\|_p = 0$ the inequality is obvious. Otherwise we have

$$\|x+y\|_p^{p - p/q} \leq \|x\|_p + \|y\|_p .$$

Since

$$p - p/q = p\left(1 - \frac{1}{q}\right) = p\left(\frac{1}{p}\right) = \underline{1}$$

we are done. $\square$

Hölder's Inequality follows from a more basic inequality, Young's inequality

Prop (Young's Inequality) Suppose $a, b \geq 0$, $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$ab \leq \frac{1}{p} a^p + \frac{1}{q} b^q.$$

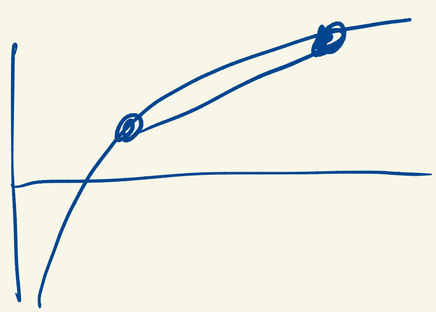
(another special case of AM-GM)

$a = \alpha^{1/p}$

$b = \beta^{1/q}$

$\alpha^{1/p} \beta^{1/q} \leq \frac{1}{p}\alpha + \frac{1}{q}\beta$

Look for something like this on HW.



$\frac{1}{2}(1+1) = 1$

$\frac{1}{p} \cdot 1 + \frac{1}{q} \cdot 1 = 1$

Exercise: Given Young's Inequality, prove Hölder's Inequality.

$x = (1, 7, -3, 6)$

$y = (4, 1, 2, 8)$

$\in \mathbb{R}^4$

$x \cdot y = \|x\| \, \|y\| \cos\theta$

$\frac{x}{\|x\|} \cdot \frac{y}{\|y\|} = \cos\theta$

$|x \cdot y| \leq \|x\|_2 \, \|y\|_2$

$\left| \frac{x}{\|x\|_2} \cdot \frac{y}{\|x\|} \right| \leq 1$

$\cos\theta$

$0 \leq \theta \leq \pi$

$-1 \leq \frac{x}{\|x\|} \cdot \frac{y}{\|y\|} \leq 1$