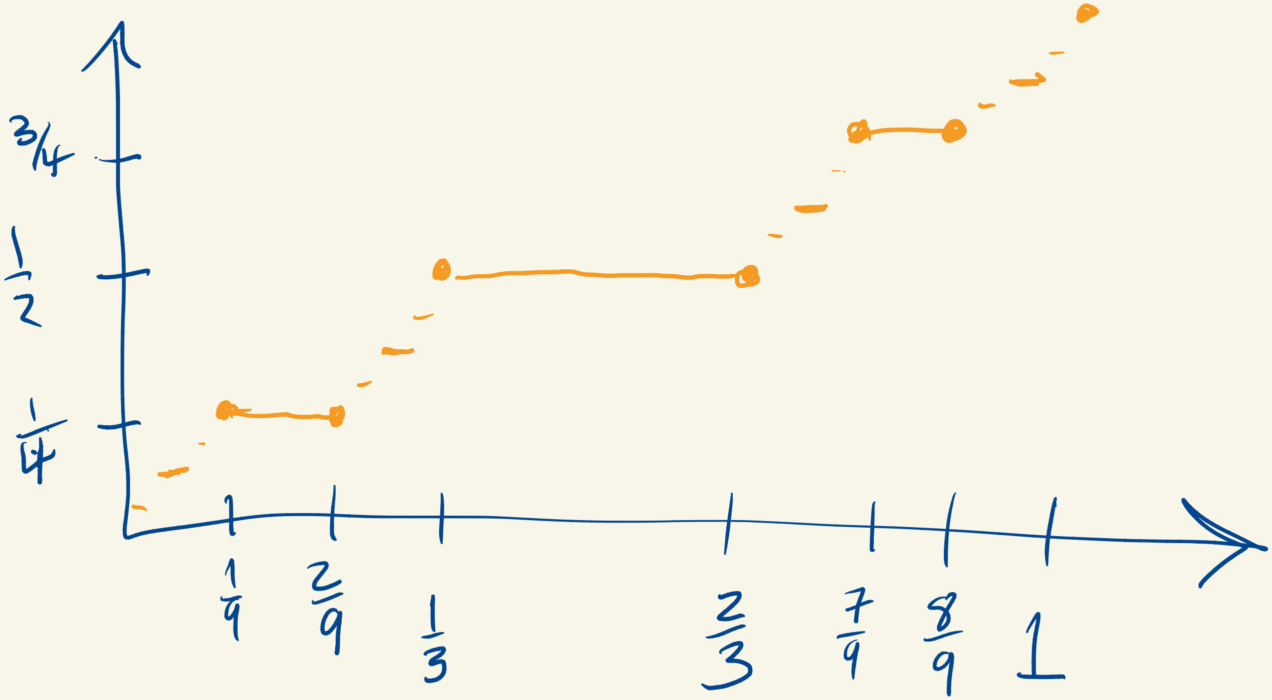
Define $F(x) = \sup\{F(z) : z \in \Delta,\ z \le x\}$



Exercise: F is increasing $(x \le y \Rightarrow F(x) \le F(y))$

Moreover $F(x_1) = F(x_2)$ with $x_1 < x_2$

$\Leftarrow$ $x_1 = 0.a_1 \cdots a_n 1 0 \cdots$ (base 3)

$x_2 = 0.a_1 \cdots a_n 2 0 \cdots$ (base 3)

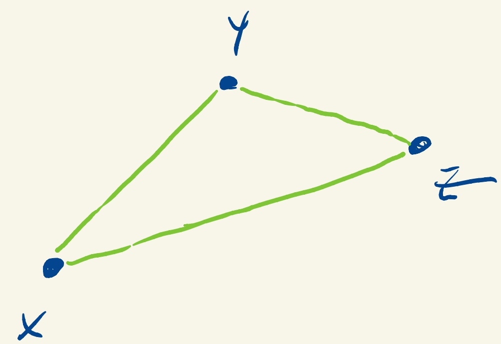
Metric Spaces:

X is a set

A metric on X is a function $d: X \times X \to \mathbb{R}$

such that

$$\left.\begin{array}{l} 1)\ d(x,y) \geq 0 \\[6pt] 2)\ d(x,y) = 0 \text{ iff } x = y \\[6pt] 3)\ d(x,y) = d(y,x) \\[6pt] 4)\ \underbrace{d(x,z) \leq d(x,y) + d(y,z)}_{\text{Triangle inequality}} \end{array}\right\} \forall\, x,y,z \in X$$



A set X equipped with a metric is a <u>metric space</u>

$$\|x\| = |x|$$

e.g. $\mathbb{R}$, $d(x,y) = |x-y|$

$\mathbb{R}^3$ $d(x,y) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + (x_3-y_3)^2}$

$\|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ Euclidean distance

$S^2 = \{x \in \mathbb{R}^3 : \underline{d(x,0) = 1}\}$

$d(x,y)$ is again Euclidean distance.

(any subset of a metric space is naturally a metric space)

$C[0,1]$, $f:[0,1] \to \mathbb{R}$, continuous

$$d(f,g) = \sup_{x \in [0,1]} |f(x) - g(x)|$$

$$\|f\|_\infty = \max_{x \in [0,1]} |f(x)| \qquad = \max_{x\in[0,1]} |f(x) - g(x)|$$

Triangle inequality. Suppose $f, g, h \in C[0,1]$

If $x \in [0,1]$ then

$$\begin{aligned} |f(x) - h(x)| &= |f(x) - g(x) + g(x) - h(x)| \\ &\le |f(x) - g(x)| + |g(x) - h(x)| \\ &\le d(f,g) + d(g,h) \end{aligned}$$

Hence $d(f,h) = \sup_{x \in [0,1]} |f(x) - h(x)| \le d(f,g) + d(g,h)$

A related concept applies to vector spaces

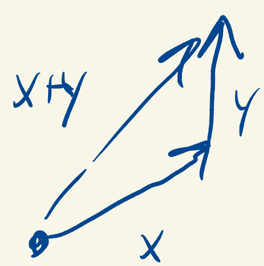
A <u>norm</u> on a vector space V is a function $\|\cdot\| : V \to R$ satisfying

1) $\|x\| \geq 0 \quad \forall x \in V$

2) $\|x\| = 0 \iff x = 0$

3) $\|\alpha x\| = |\alpha| \|x\| \quad \forall \alpha \in R,\ x \in V$ — compatability with vector space structure.

4) $\underbrace{\|x+y\| \leq \|x\| + \|y\|}_{\text{triangle inequality}}$



Given a norm on a vector space we obtain an induced metric $\quad d(x,y) = \|x-y\|$

notice $\quad d(x,0) = \|x-0\| = \|x\|$

Exercise: Show that this metric really is a metric.

Most of our previous metric constructions were of this type

Norms on $\mathbb{R}^2$

ℓ_1: $\|x\|_1 = |x_1| + |x_2|$ $\qquad x = (x_1, x_2)$

ℓ_2: $\|x\|_2 = \sqrt{x_1^2 + x_2^2}$

ℓ_∞: $\|x\|_\infty = \max(|x_1|, |x_2|)$ $\qquad \|f\|_\infty = \max\limits_{x \in [0,1]} |f(x)|$

$\max\limits_{i \in \{1,2\}} |x_i|$

Exercise: Show that the ℓ_1 and ℓ_∞ norms are norms.

We'll soon prove the ℓ_2 triangle inequality

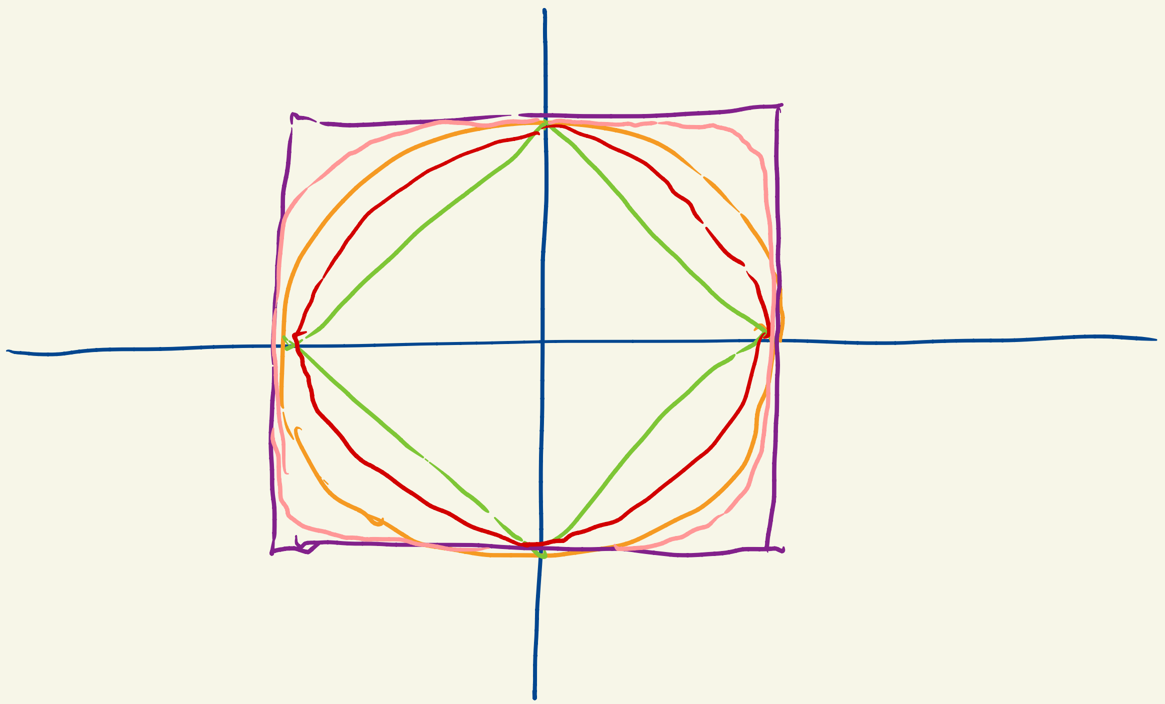
$\ell_p \quad 1 \le p < \infty$

$$\|x\|_p = \left(|x_1|^p + |x_2|^p\right)^{1/p}$$

Given a norm $\|\cdot\|$, the closed unit ball is

$$\overline{B}_1 = \{x : \|x\| \le 1\}$$

ℓ_p



$\ell_1, \ell_2, \ell_\infty$

$\sqrt{x_1^2 + x_2^2} \le 1$

$x_1^2 + x_2^2 \le 1$

$|x_1| + |x_2| \le 1$

$x_1 + x_2 \le 1$

$x_2 \le 1 - x_1$

$|x_1| \le 1$ and $|x_2| \le 1$

Exercise $\|x\|_p \longrightarrow \|x\|_\infty$ as $p \to \infty$.

fixed in $\mathbb{R}^2$

The ℓ_p norms generalize to any $\mathbb{R}^n$ $\quad \sqrt{|x_1|^2 + |x_2|^2} = \|x\|_2$

but also to certain sequences,

Def: Given $1 \le p < \infty$, ℓ_p is the set of sequences $x = (x_n)$ with

$$\left[\sum_{n=1}^{\infty} |x_n|^p\right]^{1/p} < \infty,$$

in which case $\|x\|_p =$

ℓ_∞ is the set of bounded sequences

$$\|x\|_\infty = \sup_n |x_n|$$