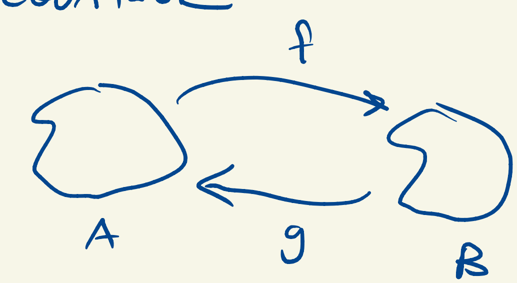
Cor: If $A \subseteq \mathbb{N}$ then A is $\underbrace{\text{at most countable}}_{\text{"countable"}}$

Cor: A set A is countable iff it has the cardinality of a subset of $\mathbb{N}$.

B, countable

Cor: If $f: A \to \mathbb{N}$ is an injection then A is countable.

Cor: If $f: A \to B$ is a surjection and A is countable then so is B.



Pf: WLOG we can assume that $A \subseteq \mathbb{N}$.

For each $b \in B$ define $g(b) = \min(\underbrace{f^{-1}(\{b\})}_{\neq \emptyset \text{ because } f \text{ is a surjection}})$.

This is well defined by the W.O.P.

Observe that $f(g(b)) = b$ and hence g is an injection from B into $\mathbb{N}$. Hence B is countable.

Cor: If $f: A \to B$ is a surjection and B is uncountable then A is uncountable.

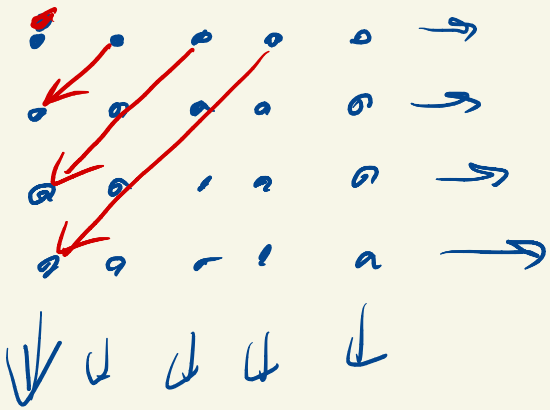
Cor: If $A \subseteq B$ and A is uncountable then so is B.

Cor: A nonempty set A is countable iff there is a surjection $\mathbb{N} \to A$.

Countable sets

1) $\mathbb{N}$

2) $\mathbb{N} \times \mathbb{N}$



3) $\mathbb{Q}_+ = \{q \in \mathbb{Q} : q > 0\}$ $\qquad \frac{a}{b}$

$f : \mathbb{N} \times \mathbb{N} \to \mathbb{Q}_+$

$(a, b) \mapsto \frac{a}{b}$

$(1, 2)$

$(2, 4)$

f is a surjection from a countable set onto $\mathbb{Q}_+$, so $\mathbb{Q}_+$ is countable.

4) $\underbrace{\mathbb{N} \times \mathbb{N} \times \mathbb{N} \cdots \times \mathbb{N}}_{k \text{ times}}$ (induction) $\quad \mathbb{N} \times \mathbb{N} \leftarrow$ base

$$(\mathbb{N} \times \cdots \times \mathbb{N}) \times \mathbb{N}$$

$$\uparrow \text{ build a surjection}$$

$$\mathbb{N} \times \mathbb{N}$$

5) $A \cup B$ where A, B are countable

trivial if A or B is empty

otherwise we can find surjections $f_A : \mathbb{N} \to A$

$f_B : \mathbb{N} \to B$

$$\mathbb{N} \times \{1, 2\} \xrightarrow{f} A \cup B$$

countable because $\subseteq \mathbb{N} \times \mathbb{N}$

$$f(n, 1) = f_A(n)$$

$$f(n, 2) = f_B(n)$$

f is a surjection

6) $\bigcup_{k=1}^{n} A_k$, each A_k countable (induction)

7) $\bigcup_{k=1}^{\infty} A_k$, each A_k countable

surjection

$$f_k : \mathbb{N} \to A_k$$

$$f : \mathbb{N} \times \mathbb{N} \to \bigcup_{k=1}^{\infty} A_k$$

$f(n,k) = f_k(n)$ is a surjection onto the union.

8) $\mathbb{Q}$ is countable

$$\mathbb{Q} = \mathbb{Q}_- \cup \{0\} \cup \mathbb{Q}_+$$

9) $\mathbb{N} \times \mathbb{N} \times \cdots$

$\{0,1\} \times \{0,1\} \times \cdots\cdots$] uncountable

The set of all sequences of 0's and 1's.

This collection is uncountable. Suppose not.

Then since the collection is clearly infinite.

Hence there is a sequence (x_n)

of sequences of 0's and 1's.

$x_n(1) = 0, 1, \quad x_1(2) = 0, 1, \quad x_1(3) = 0, 1, \cdots$

$$\begin{array}{llll} x_1 & x_1(1) & x_1(2) & x_1(3) \; \cdots \rightarrow \\ x_2 & x_2(1) & x_2(2) & x_2(3) \; \cdots \rightarrow \\ \downarrow & & & \end{array}$$

We'll build a sequence of 0's and 1's not in the list.

$$y(k) = \begin{cases} 0 & \text{if } x_k(k) = 1 \\ 1 & \text{if } x_k(k) = 0 \end{cases}$$

Since $y(k) \neq x_k(k)$ for all k, $y \neq x_k$ for all k.

This is a contradiction

Cantor's Diagonal Argument

Thm: $\mathbb{R}$ is uncountable.

Pf: It is enough to show that $[0,1]$ is uncountable.

Let x_n be a sequence in $[0,1]$.

It is enough to show that the sequence does not exhaust all of $[0,1]$.

We write

$$x_1 = 0.a_{11}\, a_{12}\, a_{13} \cdots \quad \text{(base 10)}$$

$$x_2 = 0.a_{21}\, a_{22}\, a_{23} \cdots \quad \text{(base 10)}$$

$$\vdots$$

In each case, if x_k admits two expansions we pick the terminating one.

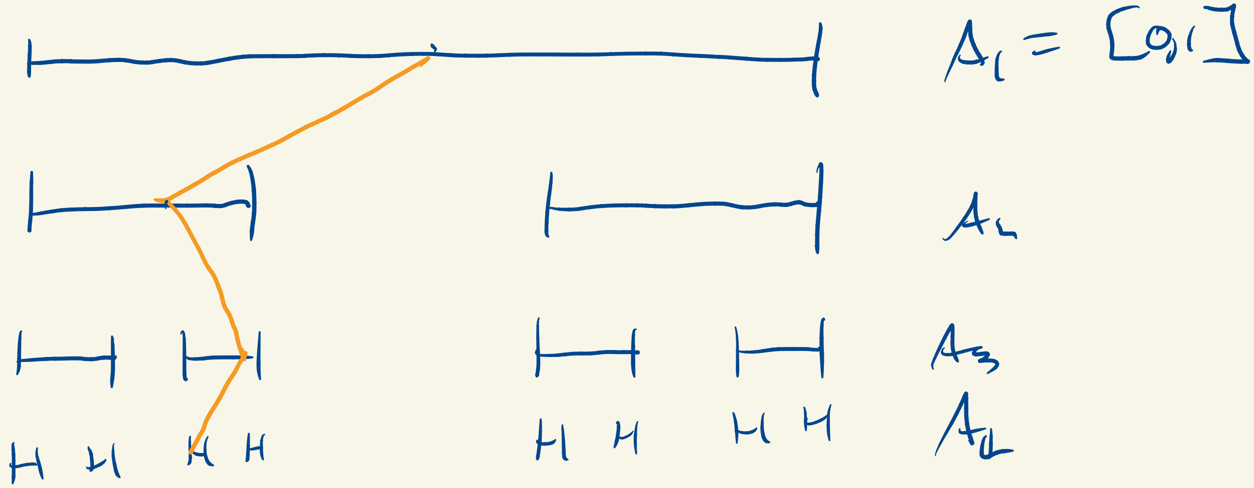
Let $b_k = \begin{cases} 7 & \text{if } a_{kk} \neq 7 \\ 3 & \text{if } a_{kk} = 7 \end{cases}$

Consider $x = 0.b_1\, b_2\, b_3 \cdots$ (base 10).

Since none of the digits of x is a 0 or a 9, x has a unique base 10 expansion.

Observe $x \neq x_1$ because $b_1 \neq a_{11}$ and because x has exactly one base 10 expansion. The same argument shows $x \neq x_k$ for all k.

Cantor Set



Each such path corresponds to a unique element of Δ

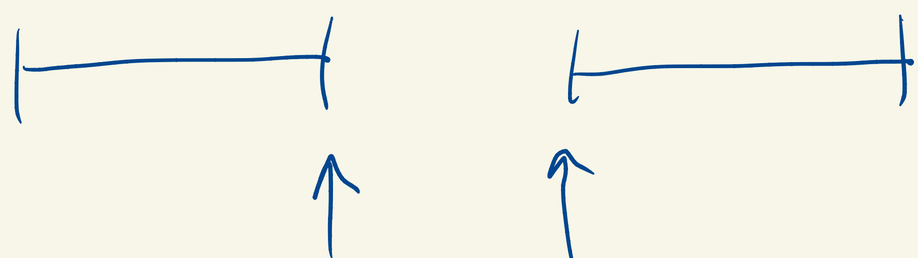
via the Nested Interval Property of $\mathbb{R}$.

We've described a bijection from $\underbrace{\{0,1\}^{\mathbb{N}}}_{\text{uncountable}}$ to $\underline{\Delta}$.

Alt: $F: \Delta \to [0,1]$, surjection.

$x \in \Delta$

$x = 0.b_1 b_2 b_3 \cdots$ (base 3) where each $b_i = 0$ or 2.



$0.02\cdots$ (base 3) $\quad 0.20\cdots$ (base 3)

~~0.1~~

$$x = \sum_{k=1}^{\infty} \frac{2a_k}{3^k} \quad \text{where} \quad a_k = 0, 1 \quad (\text{unique!})$$

$$F(x) = 0.a_1 a_2 a_3 \ldots \quad (\text{base } 2)$$

$$F(1/3) = 0.011\ldots \ (\text{base } 2) = 0.10\ldots \ (\text{base } 2) = 1/2$$

$$F(2/3) = 0.100\ldots \ (\text{base } 2) = 1/2$$

Clearly F is a surjection (from $\Delta \to [0,1]$)

↳ "Cantor function"

We can extend F from F defined on Δ to all of $[0,1]$ as follows.

First, note that F as given is monotone increasing.

Define $F(x) = \sup\{F(z) : z \in \Delta,\ z \leq x\}$