On HW: Given $x \in (0,1)$ exactly one of the following is true

a) x admits a base p expansion and the expansion does not end in a tail of all 0's or all $p-1$'s.

b) x admits exactly two expansions. One is

$$0.a_1 \cdots a_N 0 \cdots 0$$

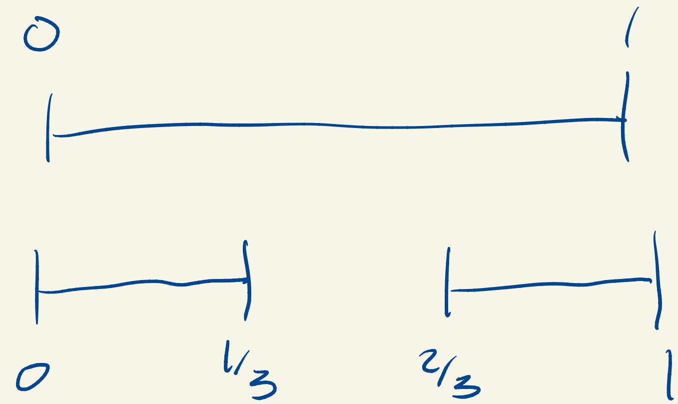
with $a_N \neq 0$ and the other is

$$0.a_1 \cdots (a_N - 1)(p-1)(p-1) \cdots$$

Cantor Set

$$A_0 = [0,1]$$

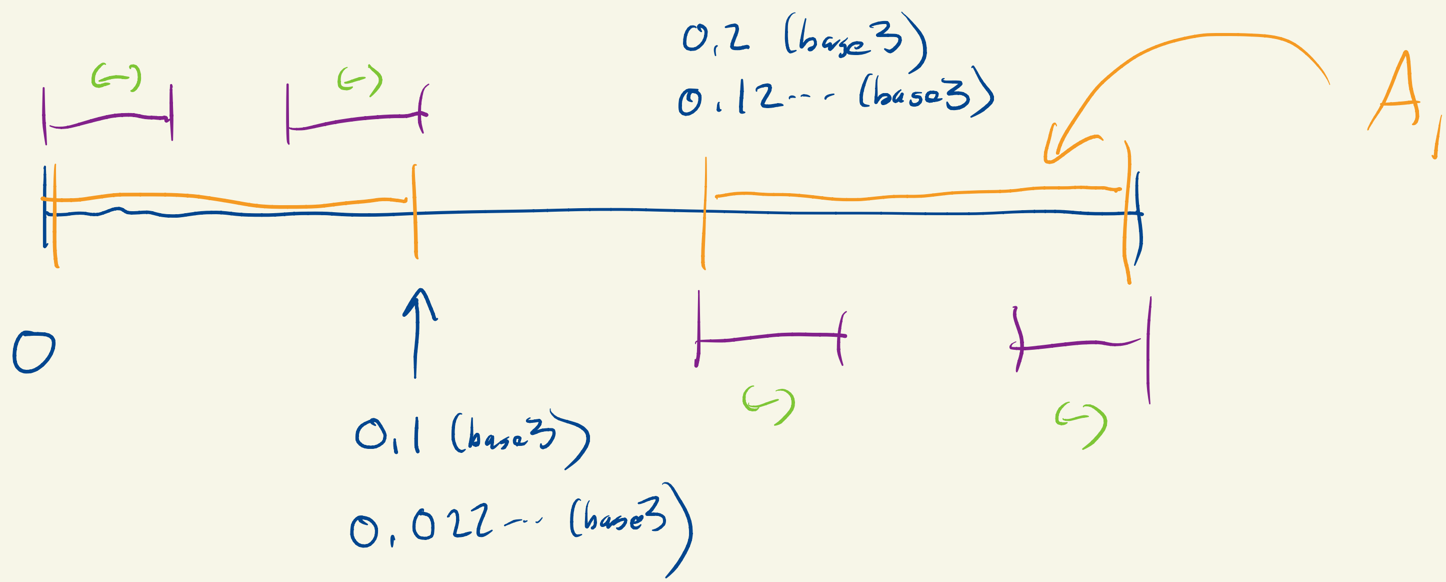
$$A_1 = [0,\tfrac{1}{3}] \cup [\tfrac{2}{3},1]$$



$A_3 =$ 

$$A_{k+1} = \frac{1}{3}A_k \cup \left(\frac{2}{3} + \frac{1}{3}A_k\right)$$

$$\Delta := \bigcap A_k$$



The elements of A_1 are exactly the elements of $[0,1]$ that admit a base 3 expansion starting with either 0 or 2.

A_2: admit a base 3 expansion where the first two digits are either 0 or 2.

Δ: admit a base 3 expansion where the digits are only 0 or 2.

Is the Cantor set big or small?

Answer 1: To construct Δ from $[0,1]$ we remove

1 interval of length $1/3$

2 intervals of length $1/3^2$

2^2 intervals of length $1/3^3$

"Total length removed"

$$\frac{1}{3} + 2\frac{1}{3^2} + 2^2\frac{1}{3^3} + 2^3\frac{1}{3^4} + \cdots$$

$$\frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^k = \frac{1}{2}\,\frac{2/3}{1-2/3} = \frac{1}{2}\,\frac{2}{3-2} = 1$$

Cantor set is small.

But the cantor set is uncountable. It is large!

Countability.

Recall sets A and B have the <u>same</u> <u>cardinality</u> (and we write $A \sim B$) if there is a bijection from A to B.

A set is *finite* if it is empty or has the cardinality of $\{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$.

A set that is not finite is infinite.

A set is *countably infinite* if it has the cardinality of $\mathbb{N}$.

A set that is infinite but not countably infinite is *uncountable*.

$\mathbb{N}$ is infinite:

1) show that every finite subset of $\mathbb{N}$ has a maximum element. (induction!)

2) Every element of $\mathbb{N}$ is not a maximum element of $\mathbb{N}$.

<u>countable</u>

$\parallel$

<u>at most countable</u> : either finite or countably infinite.

Lemma: If A is finite and $B \subseteq A$ then B is finite.

Pf: For convenience, define $S_n = \{1, \dots, n\}$.

We can assume WLOG that $A = S_n$ for some n.

The proof is obvious if $n = 1$.

Suppose the result is true for some $n \in \mathbb{N}$ and consider a set $B \subseteq S_{n+1}$. If $B \subseteq S_n$ then B is finite by the induction hypothesis. The set B is also finite if $B = \{n+1\}$.

$$1 \quad 2 \quad \cdots \quad n \;\Big|\; n+1$$

Otherwise $B \cap S_n \neq \phi$ and $n+1 \in B$.

We can construct a bijection $\phi : S_k \to B \cap S_n$ for some k and extend it to a bijection $\phi : S_{k+1} \to B$ by defining $\phi(k+1) = n+1$.

Cor: If $B \subseteq A$ and B is infinite A is infinite.

(contrapositive)

Lemma: If A is infinite and $A \subseteq \mathbb{N}$ then A is countably infinite.

Pf: Let $A_1 = A$.

Let a_1 be the least element of A_1.

(since A is nonempty, and by the Well Ordering Principle)

Let $A_2 = A_1 \setminus \{a_1\}$. Observe that A_2 is infinite (and hence nonempty!). Let a_2 be the least element of A_2.

Continuing inductively we construct a strictly monotone increasing sequence $a_1 < a_2 < \cdots$ in A,

That is, we have an injective map $\mathbb{N} \to A$.

We claim that this map is a surjection. Indeed, suppose $c \in A$. Observe that for any k, $a_k \geq k$.

In particular, $a_c \geq c$. Since $c \in A = A_{c+1} \cup \{a_1, a_2, \ldots, a_c\}$ we find that $c \in \{a_1, \ldots, a_c\}$ since each $a \in A_{c+1}$ satisfies $a > a_c \geq c$. So $c = a_k$ for some k.

Cor: If $A \subseteq \mathbb{N}$ then A is $\underbrace{\text{at most countable}}_{\text{"countable"}}$

Cor: A set A is countable iff it has the cardinality of a subset of $\mathbb{N}$.

Cor: If $f: A \to \mathbb{N}$ is an injection then A is countable.