Last class: (x_n)

M is an eventual upper bound if there exists N

so if $n \geq N$, $x_n \leq M$.

$$\limsup_{n \to \infty} x_n = \inf \{ M : M \text{ is an e.u.b.} \}$$

e.g. x_n is an enumeration of $\mathbb{Q} \cap [0,1]$

$$\limsup_{n \to \infty} x_n = 1$$

Alternative formulation (x_n)

$$T_N = \sup \{ x_N, x_{N+1}, x_{N+2}, \ldots \}$$

$$= \sup_{n \geq N} x_n$$

$$T_{N+1} \leq T_N$$

(T_N) is monotone decreasing!

(it converges to a limit, possibly $-\infty$)

$$\overline{\lim_{n\to\infty}} x_n = \lim_{N\to\infty} T_N = \lim_{N\to\infty} \sup_{n\geq N} x_n$$

$$= \inf_{N\geq 1} T_N = \inf_{N \geq 1} \sup_{n \geq N} x_n$$

Claim: $\overline{\lim_{n\to\infty}} x_n = \limsup_{n\to\infty} x_n$

Observe that if $n \geq N$ then $x_n \leq T_N$.

Hence each T_N is an eventual upper bound for the sequence.

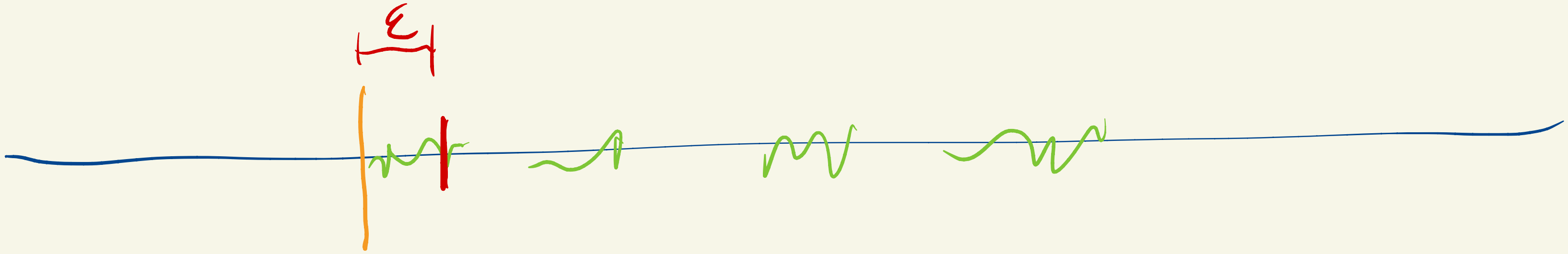
Hence

$$\overline{\lim_{n\to\infty}}\, x_n \geq \limsup_{n\to\infty} x_n\,.$$

Conversely, suppose for the moment that

$\limsup\limits_{n\to\infty} x_n = M \in \mathbb{R}$. Let $\varepsilon > 0$. Then there

exists an e.u.b. B such that $B < M + \varepsilon$.



Hence there exists N such that if $n \geq N$ then T_N

$x_n \leq B$. Recall $T_N = \sup\limits_{n \geq N} x_n$. Hence $T_N \leq B$.

Consequently $T_N < M + \varepsilon$. So $\overline{\lim\limits_{n\to\infty}}\, x_n = \inf\limits_N T_N < M + \varepsilon$.

This is true for all $\varepsilon > 0$. So $\overline{\lim\limits_{n\to\infty}}\, x_n \leq M = \limsup\limits_{n\to\infty} x_n$.

The same inequality when $\limsup\limits_{n\to\infty} x_n = +\infty$ is obvious and

the case $-\infty$ is on HW.

lim inf: m is an eventual lower bound for (x_n) if there exists N such that if $n \geq N$ then $m \leq x_n$.

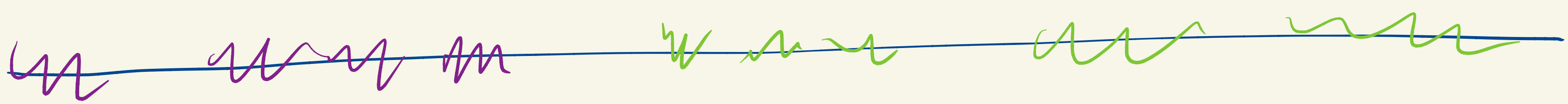
$$\liminf_{n\to\infty} x_n = \begin{cases} \sup\{m : m \text{ is an e.l.b.}\} \\ \sup_{N\geq 1} \inf_{n\geq N} x_n \\ \lim_{N\to\infty} \inf_{n\geq N} x_n \\ -\limsup_{n\to\infty}(-x_n) \end{cases}$$

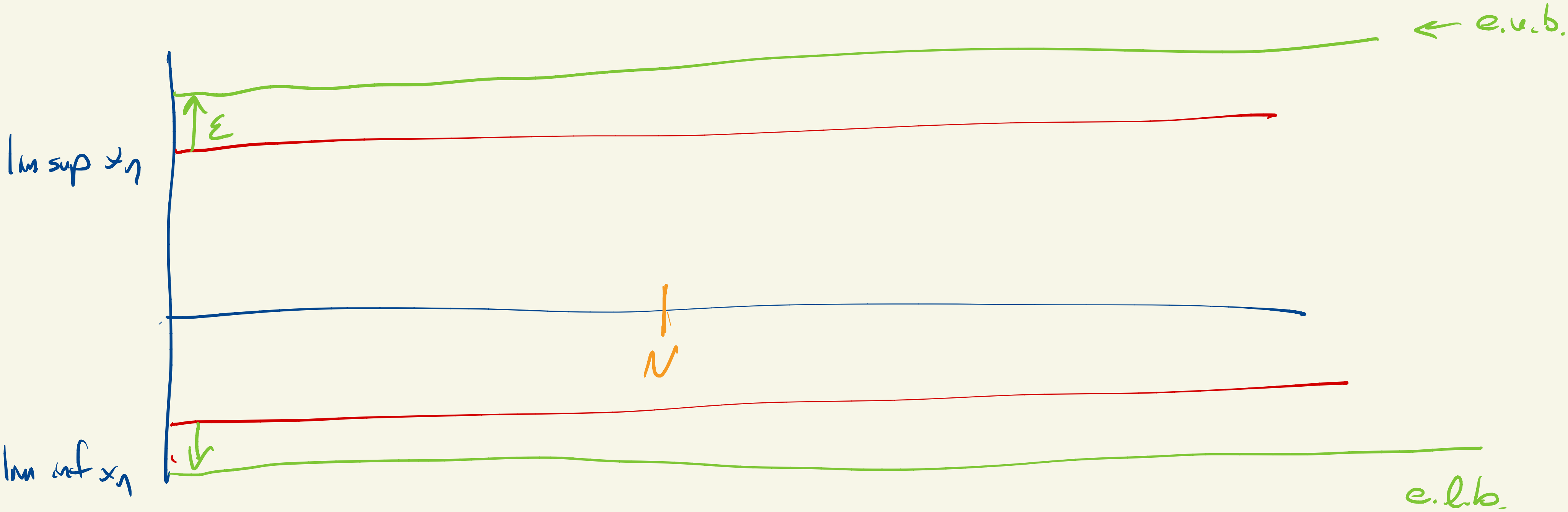
Lemma: $\liminf_{n\to\infty} x_n \le \limsup_{n\to\infty} x_n$.

Pf: Let m and M be an eventual lower and upper bound respectively for the sequence. Hence there exists N such that $$m \le x_N \le M.$$

Hence $m \le M$. Recall $\liminf x_n = \sup\{m : m \text{ is an e.l.b}\}$.

Hence $\liminf x_n \le M$. Hence $\liminf x_n \le \limsup x_n$. $\square$





Exercise: $\lim_{n\to\infty} x_n = L$ iff $\limsup_{n\to\infty} x_n = L = \liminf_{n\to\infty} x_n$

$(L \in \mathbb{R}$ or $L = \infty$ or $L = -\infty)$

Base p expansions

Let $p \in \mathbb{N}_{\geq 2} = \{ n \in \mathbb{N} : n \geq 2 \}$

$$\mathcal{D}_p = \{ 0, 1, \dots, p-1 \} \qquad \mathcal{D} \quad \text{"digits"}$$

Given $(a_k)_{k=1}^{\infty}$ $a_k \in \mathcal{D}_p$ we define

$$\sum_{k=1}^{\infty} a^k = \frac{a}{1-a}$$

$$0.a_1 a_2 a_3 \dots \text{ (base } p) = \sum_{k=1}^{\infty} \frac{a_k}{p^k}$$

$$\frac{1}{1-a}$$

Does this series converge? $\displaystyle \sum_{k=1}^{\infty} \frac{1}{p^k} = \sum_{k=1}^{\infty} \left(\frac{1}{p}\right)^k$

$$(1-a)(1+a+a^2+\cdots+a^N) = 1-a^{N+1}$$

Lemma: The series $\sum_{k=1}^{\infty} \frac{a_k}{p^k}$ with each $a_k \in \mathcal{D}_p$

converges to a number in $[0,1]$.

Pf: Since each term is non negative we can employ the comparison test.

Each $\frac{a_k}{p^k} \leq \frac{p-1}{p^k}$.

Observe $\sum_{k=1}^{\infty} \frac{p-1}{p^k} = (p-1) \sum_{k=1}^{\infty} \left(\frac{1}{p}\right)^k = (p-1) \frac{\frac{1}{p}}{1-1/p}$

$$= (p-1)\,\frac{1}{p-1}$$

$$= 1.$$

Hence $0 \le \sum_{k=1}^{\infty} \frac{a_k}{p^k} \le 1.$ $\square$

Question: given $x \in [0,1]$, does it admit a base p expansion? How many?

$0.5 \quad 0.499\ldots$

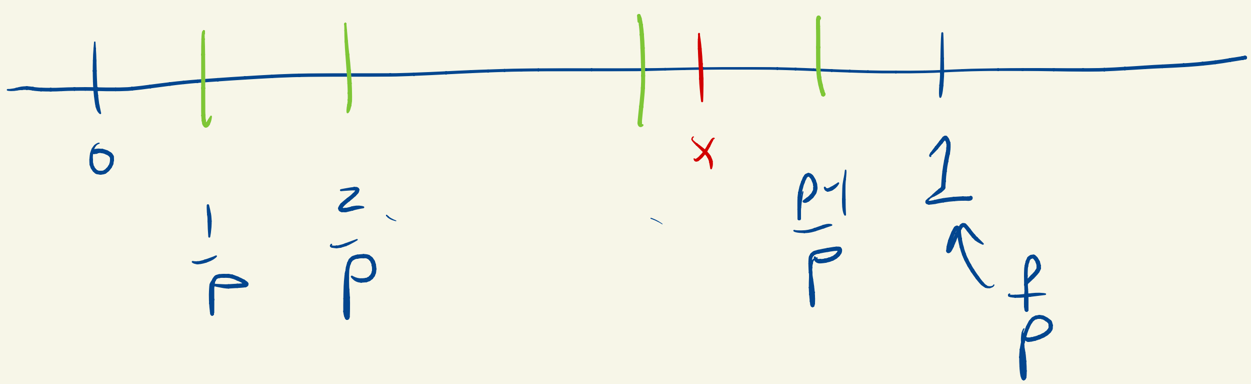
Prop: Each $x \in [0,1]$ admits a base p expansion.

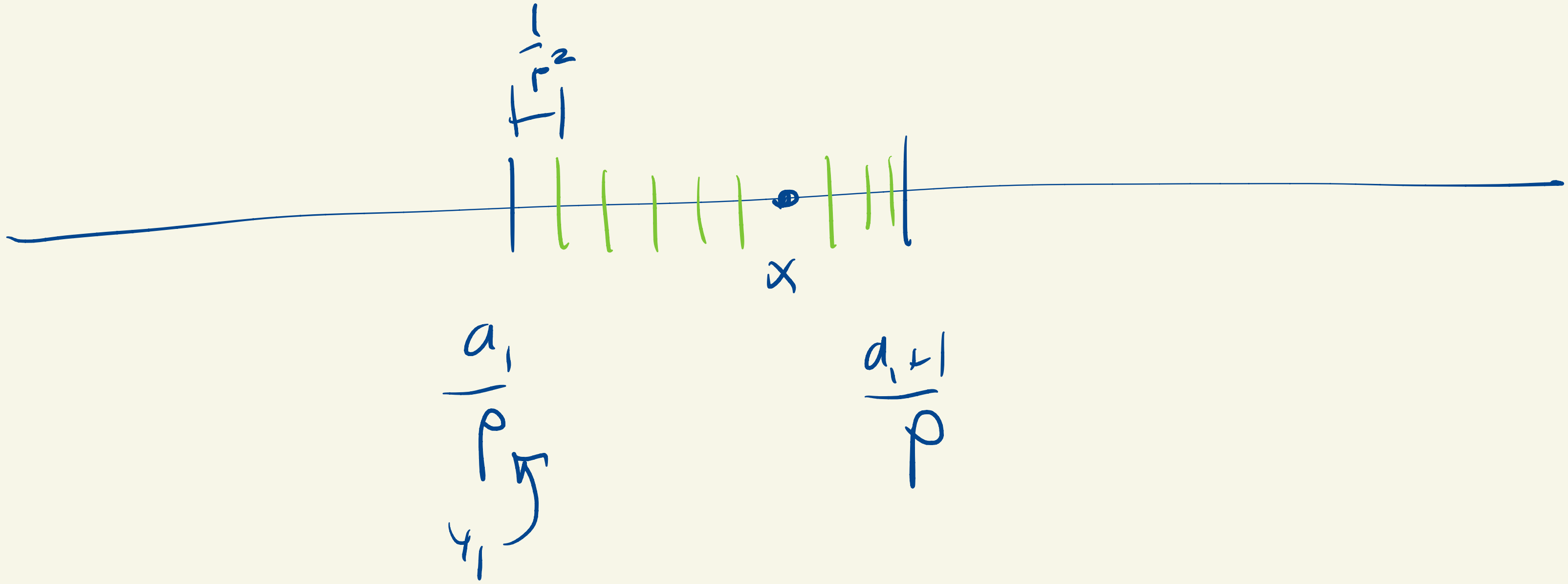
Pf: The case $x = 0$ is trivial.

Suppose $0 < x \leq 1$.

Let $a_1 = \max \left\{ d \in \mathbb{N}_{\geq 0} : \frac{d}{p} < x \right\}$ and

observe that $a_1 \in \mathcal{D}_p$.





Let $y_1 = \frac{a_1}{p}$ and observe $y_1 < x \le y_1 + \frac{1}{p}$

Let $a_2 = \max \left\{ d \in \mathbb{N}_{\geq 0} : y_1 + \frac{d}{p^2} < x \right\}$

Let $y_2 = y_1 + \frac{a_2}{p^2}$

Continuing inductively we can find $a_1, a_2, \ldots \in \mathbb{Q}_p$

such that $y_n := \frac{a_1}{p} + \frac{a_2}{p^2} + \cdots + \frac{a_n}{p^n}$ satisfies

$$y_n < x \leq y_n + \frac{1}{p^n}$$

Notice that $|x - y_n| \leq \frac{1}{p^n}$.

By the squeeze theorem, $x - y_n \to 0$ or $y_n \to x$.

Hence $\sum_{k=1}^{\infty} \frac{a_k}{p^k} = x$.