How is $\mathbb{R}$ different from $\mathbb{Q}$?

Completeness:

Axiom of Completeness:

Every nonempty subset of $\mathbb{R}$ that is bounded above admits a supremum.

If $A \subseteq \mathbb{R}$ we say b is a supremum of A if

1) b is an upper bound for A
(for all $a \in A$, $a \leq b$).

2) If b' is any upper bound for A, $b \leq b'$.
(leastness)

Manifestations:

1) Cauchy criterion (Cauchy sequences converge)

2) Bolzano - Weierstrass (bounded sequences have convergent subsequences)

3) Montone Convergence Thm (monotone bounded sequences converge)

4) Nested interval property (any sequence of nested closed intervals has non empty intersection)

Extended Real Numbers:

$$\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$$

Rules

1) $\infty \geq x$ for all $x \in \overline{\mathbb{R}}$

2) $-\infty \leq x$ for all $x \in \overline{\mathbb{R}}$

3) $x + \infty = \infty$ unless $x = -\infty$

4) $-\infty + x = -\infty$ unless $x = \infty$

5) $\infty + (-\infty)$ is not defined

6)
$$a \cdot \infty = \begin{cases} \infty & \text{if } a > 0 \quad (a = \infty \text{ also!}) \\ -\infty & \text{if } a < 0 \\ \text{undefined} & \text{if } a = 0 \end{cases}$$

If we work with the extended real numbers, we can take inf / sup of any set.

If $A \subseteq \mathbb{R}$ and is not bounded above

$\sup A = \infty$.

$\sup \emptyset = -\infty$.

Recall $\lim_{n\to\infty} x_n = L$ if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$ $|L - x_n| < \varepsilon$.

Limits need not exist.

$x_n = n$

$x_n = (-1)^n$

$x_n = r_n$ where (r_n) is an enumeration of $\mathbb{Q} \cap [0,1]$

We have, however, two related objects,

limit infimum (lim inf)

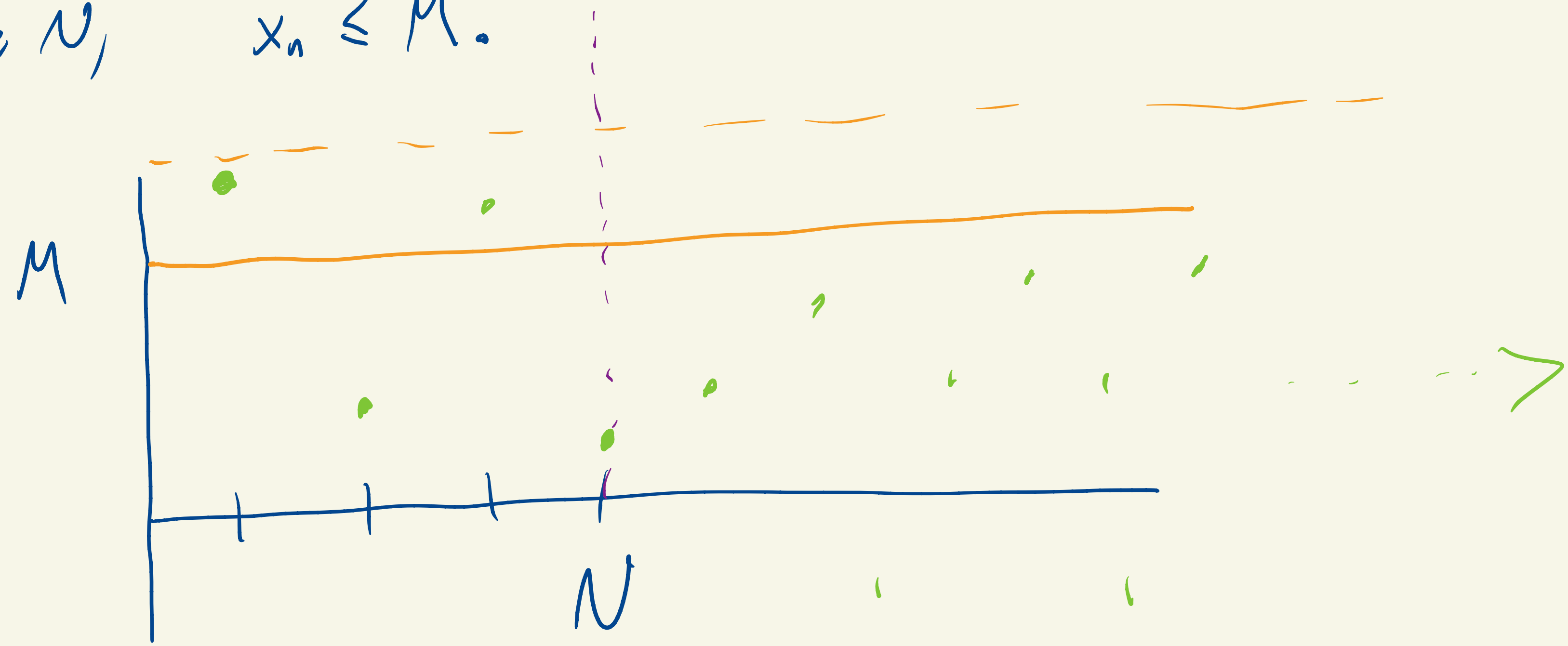
limit supremum (lim sup)

that always exist.

lim sup

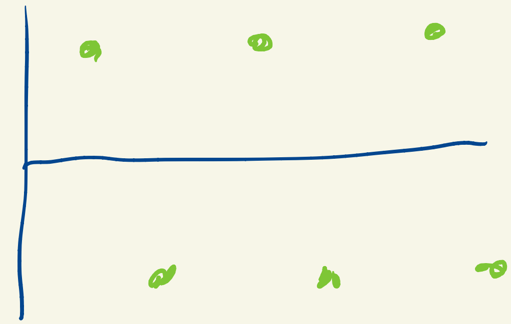
Let (x_n) be a sequence.

We say $M \in \overline{\mathbb{R}}$ is an eventual upper bound for the sequence if there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n \leq M$.



Def: $\limsup_{n\to\infty} x_n = \inf\{M : M \text{ is an e.u.b. for } (x_n)\}$

e.g. $x_n = (-1)^n$ $\quad \limsup_{n\to\infty} x_n = 1$



If $M \geq 1$ then it is an eventual upper bound as it is an upper bound for the sequence.

Suppose $M < 1$. Given any $N \in \mathbb{N}$, either x_N or $x_{N+1} = 1 > M$. So M is not an e.u.b. So the set of e.u.b's for the sequence is $[1, \infty)$ which has 1 as an inf.

$x_n = \frac{1}{n}$

$\limsup_{n\to\infty} \frac{1}{n} = 0$

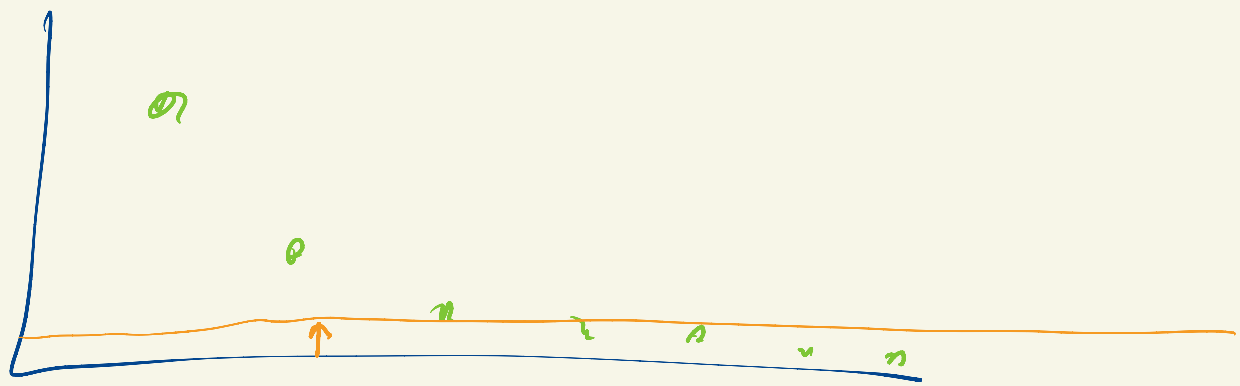
If $M \leq 0$ it is not an e.u.b.

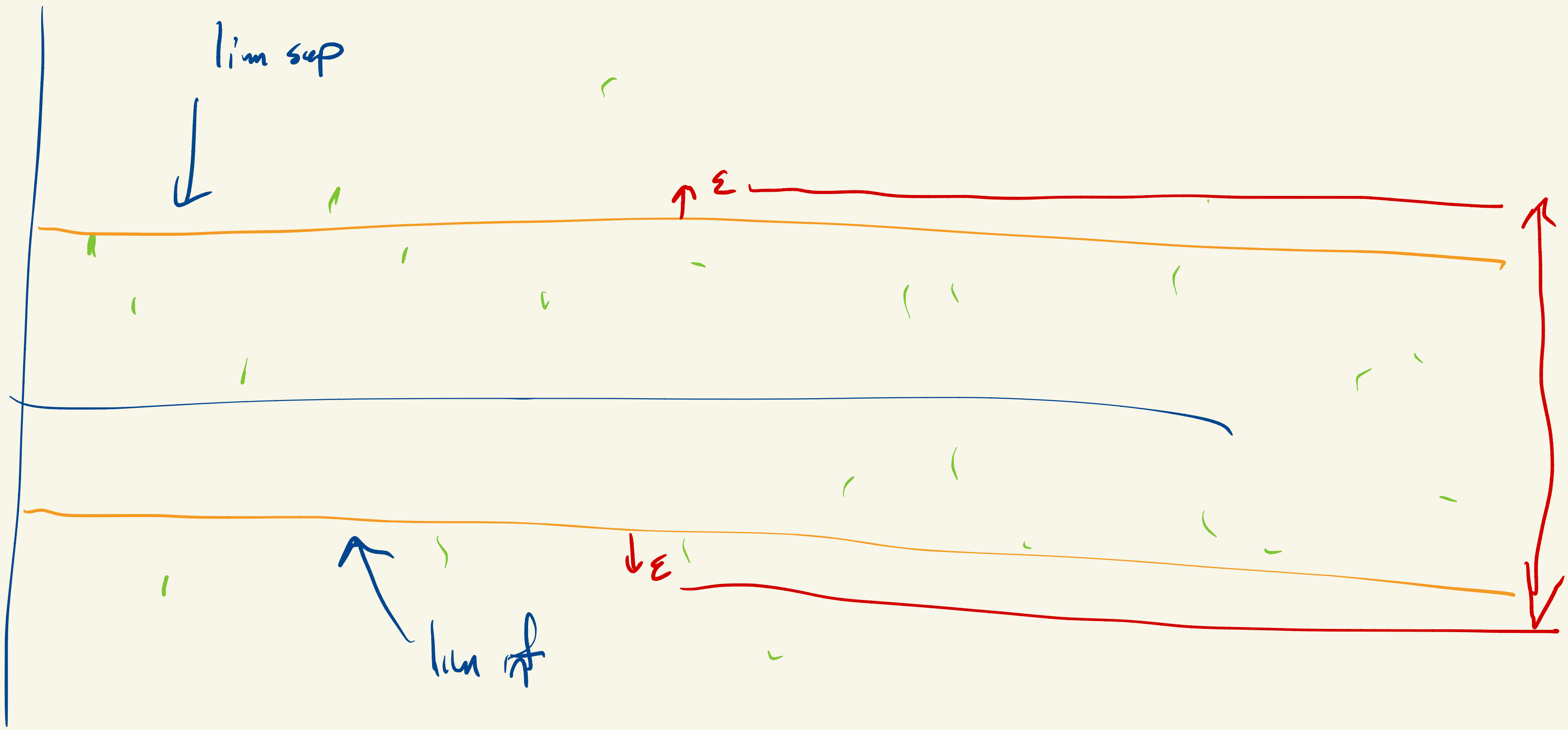
Suppose $M > 0$. Pick $N \in \mathbb{N}$ with $\frac{1}{N} < M$.

Then if $n \geq N$, $\frac{1}{n} \leq \frac{1}{N} \leq M$. So M is

an e.u.b. So the set of e.u.b's is $(0, \infty]$.

So $\limsup_{n \to \infty} \frac{1}{n} = \inf (0, \infty] = 0$.




lim sup
ε
ε
lim inf