### Integration of Step Functions

A **partition** $\mathcal{P}$ of $[a, b]$ is a collection $\{x_k\}_{k=0}^n$ such that

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

More succinctly, a partition is a finite subset of $[a, b]$ containing $a$ and $b$. It is helpful to think of a partition as dividing $[a, b]$ into intervals $I_k = [x_{k-1}, x_k]$, each of which having length $dx_k = x_k - x_{k-1}$. A partition $\mathcal{P}'$ is said to be **finer** than $\mathcal{P}$ if $\mathcal{P}' \supseteq \mathcal{P}$. Given any two partitions $\mathcal{P}$ and $\mathcal{P}'$, the **common refinement** of the partitions is $\mathcal{P} \cup \mathcal{P}'$.

A **step function** is a function $g : [a, b] \to \mathbb{R}$ such that there exists a partition $\mathcal{P}$ for which $g$ is constant on each open interval $(x_{k-1}, x_k)$ of the partition. Any partition that satisfies this condition for $g$ will be called a **step partition** (for $g$). Clearly every refinement of a step partition for $g$ is also a step partition for $g$. We use the notation $\mathrm{Step}[a, b]$ for the set of all step functions on $[a, b]$.

Thinking of integration as measuring signed area under the graph of function, it is relatively straightforward to integrate step functions. If $g$ is a step function with step partition $\mathcal{P}$, and if $g(x) = g_k$ on $(x_{k-1}, x_k)$, we define

$$\int_a^b g = \sum_{k=1}^n g_k dx_k.$$

One needs to verify, however, that this definition of integral does not depend on the choice of step partition. That is, if $\mathcal{P}$ and $\mathcal{P}'$ are two step partitions for $g$, it must hold that

$$\sum_{k=1}^n g_k dx_k = \sum_{k=1}^{n'} g'_k dx'_k. \tag{1}$$

By reducing to a common refinement, it is enough to show that (1) holds when $\mathcal{P}'$ is a refinement of $\mathcal{P}$.

**Exercise 1:** Establish (1) when $\mathcal{P}'$ is a refinement of $\mathcal{P}$. First assume that $\mathcal{P}' = \mathcal{P} \cup \{x'\}$, and then deduce the result for a general refinement by induction.

### Properties of the Integral

Notice that the set of step functions on $[a, b]$ is a vector subspace of $B[a, b]$. Indeed, if $f$ and $g$ are step functions with step partitions $\mathcal{P}_f$ and $\mathcal{P}_g$, let $\mathcal{P}$ be the common refinement (so $\mathcal{P}$ is a step partition for $f$ and for $g$). Then on each interval $(x_{k-1}, x_k)$ we have $f(x) = f_k$ and $g(x) = g_k$ and

$$(f + g)(x) = f_k + g_k.$$

Hence $\mathcal{P}$ is a step partition for $f + g$. Similarly, if $f$ is step function, then so is $cf$ for any $c \in \mathbb{R}$.

One of the most important properties of the integral is that the map taking $f$ to $\int_a^b f$ is linear.

**Theorem 1:** Let $f$ and $g$ be step functions on $[a,b]$. Then

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g,$$

and for any $\alpha \in \mathbb{R}$,

$$\int_a^b \alpha f = \alpha \int_a^b f.$$

**Exercise 2:** Prove Theorem 1 directly from the definition of the integral of step functions.

You may have wondered when you were first introduced to integration why it was defined in terms of *signed* area under the graph of a function. One mathematical motivation is that it ensures that the integral is linear.

Another elementary property of the integral is its montonicity (which also relies on the signed area interpretation of the integral).

**Theorem 2:** Let $f$ and $g$ be step functions on $[a,b]$ such that $f(x) \geq g(x)$ for every $x \in [a,b]$. (That is, $f \geq g$.) Then

$$\int_a^b f \geq \int_a^b g.$$

*Proof.* Let $\mathcal{P}$ be a step partition for $f$ and $g$. Then for each $k$ we have $f_k \geq g_k$ and hence

$$\int_a^b f = \sum_{k=1}^n f_k dx_k \geq \sum_{k=1}^n g_k dx_k = \int_a^b g.$$

□

Suppose $f$ is a step function. Any step partition for $f$ is also a step partition for $|f|$ and hence $|f|$ is also a step function. Moreover, for any $x \in [a,b]$ we have

$$-|f(x)| \leq f(x) \leq |f(x)|$$

and hence by Theorem 1 and Theorem 2 we have

$$-\int_a^b |f| \leq \int_a^b f \leq \int_a^b |f|.$$

We have therefore established the following estimate, which can be thought of as a relationship between the signed area $\int_a^b f$ and the unsigned area $\int_a^b |f|$.

**Theorem 3:** Let $f$ be a step function on $[a,b]$. Then

$$\left|\int_a^b f\right| \leq \int_a^b |f|.$$

One final property of the integral is that it can be computed by breaking the domain up into pieces and computing the integral on each piece.

**Theorem 4:** Suppose $f$ is a step function on $[a,b]$ and suppose $a \le c \le b$. Then

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

*Proof.* Let $\mathcal{P}$ be any step partition for $f$. Without loss of generality, we can assume that $x_N = c$ for some $N$ (otherwise, we can consider a finer partition by adding the point $c$). Then

$$\int_a^b f = \sum_{k=1}^{n} f_k dx_k = \sum_{k=1}^{N} f_k dx_k + \sum_{k=N+1}^{n} f_k dx_k = \int_a^c f + \int_c^b f.$$

□

**Riemann Integrable Functions**

We would like to extend the definition of the integral to a broader class of functions than step functions. Given a function $f \in B[a,b]$, we would like to define an integral for $f$ that preserves the properties of Theorems 1, 2, 3 and 4 of the integral for step functions. Although we will not be able to do this for all functions in $B[a,b]$, we will be able to do so for a large class of functions.

Given a function $f \in B[a,b]$, and step functions $g$ and $G$ with $g \le f \le G$, we would want to have

$$\int_a^b g \le \int_a^b f \le \int_a^b G.$$

We define the **upper Riemann integral** of $f$ to be

$$\overline{\int_a^b} f = \inf_{\substack{G \in \mathrm{Step}[a,b] \\ G \ge f}} \int_a^b G$$

and the **lower Reimann integral** of $f$ to be

$$\underline{\int_a^b} f = \sup_{\substack{g \in \mathrm{Step}[a,b] \\ g \le f}} \int_a^b g.$$

For any $f \in B[a,b]$, it is an immediate consequence of the definition that

$$\underline{\int_a^b} f \le \overline{\int_a^b} f.$$

Moreover, if $f$ is a step function, then

$$\begin{aligned}\underline{\int_a^b} f &= \sup_{\substack{g \in \mathrm{Step}[a,b] \\ g \le f}} \int_a^b g \\ &\ge \int_a^b f \\ &\ge \inf_{\substack{G \in \mathrm{Step}[a,b] \\ G \ge f}} \int_a^b G \\ &= \overline{\int_a^b} f.\end{aligned}$$

Hence if $f$ is a step function, then

$$\overline{\int_a^b} f = \underline{\int_a^b} f = \int_a^b f.$$

The class of function for which we have equality of the upper and lower Riemann integrals is known as the set of **Riemann integrable functions**, $\mathcal{R}[a,b]$. If $f \in \mathcal{R}[a,b]$, then we define

$$\int_a^b f = \overline{\int_a^b} f \left( = \underline{\int_a^b} f \right).$$

We have just shown therefore that $\mathrm{Step}[a,b] \subseteq \mathcal{R}[a,b]$, and that the Riemann integral of a step function agrees with the integral we have already defined for step functions.

It is perhaps surprising that not every function in $B[a,b]$ is Riemann integrable. An example of such a function is given by $\chi_{\mathbb{Q}}$.

**Exercise 3:** Prove that $\overline{\int_0^1} \chi_{\mathbb{Q}} = 1$ but $\underline{\int_0^1} \chi_{\mathbb{Q}} = 0$.

**Upper and Lower Riemann Sums**

Sometimes it is convenient to describe the upper and lower Riemann integrals of a function in terms of limits of certain near optimal step functions.

Let $f \in B[a,b]$ and let $\mathcal{P}$ be a partition of $[a,b]$. Given this partition, we define $M_k = \sup_{x \in [x_{k-1}, x_k]} f(x)$ and $m_k = \inf_{x \in [x_{k-1}, x_k]} f(x)$. We then associate with $f$ and $\mathcal{P}$ step functions $\overline{f}_{\mathcal{P}}$ and $\underline{f}_{\mathcal{P}}$ that are equal to $M_k$ and $m_k$ respectively on $(x_{k-1}, x_k)$ and are equal to $f(x_k)$ for each $k$. The **upper Riemann sum** (for the function $f$ and partition $\mathcal{P}$) is

$$U(f, \mathcal{P}) = \int_a^b \overline{f}_{\mathcal{P}} = \sum_{k=1}^n M_k dx_k$$

and the **lower Riemann sum**

$$L(f, \mathcal{P}) = \int_a^b \underline{f}_{\mathcal{P}} = \sum_{k=1}^n m_k dx_k.$$

**Exercise 4:** Given a function $f \in B[a,b]$ and a step function $G \geq f$, show that for any $\epsilon > 0$ there exists a partition $\mathcal{P}'$ such that $\epsilon + \int_a^b G \geq U(f, \mathcal{P}')$ Prove a similar result for step functions $g \leq f$. Conclude that

$$\overline{\int_a^b} f = \inf_{\mathcal{P}} U(f, \mathcal{P})$$

and

$$\underline{\int_a^b} f = \sup_{\mathcal{P}} L(f, \mathcal{P}).$$

**Characterization of Riemann Integrable Functions**

Given the definition of Riemann integrability, it is not necessarily easy to determine whether a given function is Riemann integrable. On the face of things, one would have to compute the upper and lower Riemann integrals, and then verify that they are the same. The following result helps identify Riemann integrable functions without having to compute upper and lower Riemann integrals.

**Proposition 5:** Let $f \in B[a,b]$. Then the following are equivalent.

1. $f \in \mathcal{R}[a,b]$.
2. For any $\epsilon > 0$ there exist step functions $g$ and $G$ with $g \leq f \leq G$ and such that
$$\int_a^b (G - g) < \epsilon.$$
3. For any $\epsilon > 0$ there exists a partition $\mathcal{P}$ such that
$$U(f, \mathcal{P}) < L(f, \mathcal{P}) + \epsilon.$$

*Proof.* Suppose $f$ is Riemann integrable. Let $G$ be a step function such that $G \geq f$ and

$$\int_a^b G < \overline{\int_a^b} f + \epsilon/2.$$

Let $g$ be a step function such that $g \leq f$ and

$$\int_a^b g > \underline{\int_a^b} f - \epsilon/2.$$

Then

$$\int_a^b G < \overline{\int_a^b} f + \epsilon/2 = \underline{\int_a^b} f + \epsilon/2 < \int_a^b g + \epsilon.$$

So

$$\int_a^b (G - g) < \epsilon.$$

Conversely, suppose $f$ is not Riemann integrable. Let $\epsilon = \overline{\int_a^b} f - \underline{\int_a^b} f$. Then for any step functions $g$ and $G$ with $g \le f \le G$ we have

$$\int_a^b g \le \underline{\int_a^b} f = \overline{\int_a^b} f - \epsilon \le \int_a^b G - \epsilon.$$

Hence

$$\int_a^b (G - g) \ge \epsilon$$

for all step functions $G$ and $g$ with $g \le f \le G$.

The equivalence of statements 2 and 3 is left for the reader. □

**Exercise 5:** Prove the equivalence of statements 2 and 3 in Proposition 5.

One important application of Proposition 5 is that it allows for an easy proof that $C[a,b] \subseteq \mathcal{R}[a,b]$.

**Theorem 6:** $C[a,b] \subseteq \mathcal{R}[a,b]$.

*Proof.* Let $f \in C[a,b]$. Let $\epsilon > 0$. Since $f$ is uniformly continuous, there exists a $\delta > 0$ such that if $|x - z| < \delta$, then $|f(x) - f(z)| < \epsilon/(b-a)$. Pick $N \in \mathbb{N}$ such that $(b-a)/N < \delta$, and let $\mathcal{P}$ be the partition $\{a + k(b-a)/N : 0 \le k \le N\}$. For each $k$, we define

$$G_k = \sup_{x \in I_k} f(x) \qquad g_k = \inf_{x \in I_k} f(x).$$

Note that $0 \le G_k - g_k \le \epsilon/(b-a)$ for each $k$.

Let $G$ be the step function that equals $G_k$ on the interior of $I_k$ and that equals $f$ at each $x_k$, and define $g$ similarly. Then $g \le f \le G$ on $[a,b]$ and moreover

$$\int_a^b G - g = \sum_{k=1}^{N} (G_k - g_k)\frac{b-a}{N} < \frac{\epsilon}{b-a}\frac{b-a}{N} N = \epsilon.$$

So Proposition 5 implies $f$ is Riemann integrable. □

### Properties of the Integral

We would like to extend Theorems 1, 2, 3 and 4 to all of $\mathcal{R}[a,b]$. The extension of Theorem 2 is immediate from the definition.

**Theorem 7:** Suppose $f, g \in \mathcal{R}[a,b]$ with $f \le g$. Then

$$\int_a^b f \le \int_a^b g.$$

*Proof.* For any step function $H$ with $g \leq H$ we have $f \leq H$ as well and hence

$$\int_a^b f = \overline{\int_a^b} f = \inf_{\substack{H \in \mathrm{Step}[a,b] \\ H \geq f}} \int_a^b H \leq \inf_{\substack{H \in \mathrm{Step}[a,b] \\ H \geq g}} \int_a^b H = \overline{\int_a^b} g = \int_a^b g.$$

□

To establish the linearity of the integral we need to work a little harder.

**Theorem 8:** Suppose $f, g \in \mathcal{R}[a,b]$. Then $f + g \in \mathcal{R}[a,b]$ and

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g. \tag{2}$$

Also, for every $\alpha \in \mathbb{R}$, $\alpha f \in \mathcal{R}[a,b]$ and

$$\int_a^b \alpha f = \alpha \int_a^b f. \tag{3}$$

*Proof.* Let $f, g \in \mathcal{R}$. For each $n \in \mathbb{N}$, let $h_{f,n}$ and $H_{f,n}$ be step functions with $h_{f,n} \leq f \leq H_{f,n}$ and with

$$\int_a^b f - \frac{1}{n} \leq \int_a^b h_{f,n} \leq \int_a^b f \leq \int_a^b H_{f,n} \leq \int_a^b f + \frac{1}{n}.$$

The existence of these step functions follow from the Riemann integrability of $f$. Let $h_{g,n}$ and $H_{g,n}$ be a similar sequence of step functions for $g$.

Notice that for each $n$, $H_{f,n} + H_{g,n} \geq f + g$. Hence for each $n$,

$$\overline{\int_a^b} (f+g) \leq \int_a^b (H_{f,n} + H_{g,n}) = \int_a^b H_{f,n} + \int_a^b H_{g,n}.$$

Taking the limit in $n$ we conclude

$$\overline{\int_a^b} f + g \leq \int_a^b f + \int_a^b g.$$

A similar argument with the sequences $(h_{f,n})$ and $(h_{f,n})$ yields the inequality

$$\int_a^b f + \int_a^b g \leq \underline{\int_a^b} f + g.$$

However, it is always true that $\overline{\int_a^b} (f+g) \geq \underline{\int_a^b} (f+g)$, so we conclude that

$$\overline{\int_a^b} (f+g) = \underline{\int_a^b} (f+g) = \int_a^b f + \int_a^b g.$$

Hence $(f+g)$ is integrable and equation (2) holds.

The proof that $\alpha f$ is integrable and that (3) holds is left as an exercise. □

**Exercise 6:** Suppose $f \in \mathcal{R}[a,b]$ and $\alpha \in \mathbb{R}$. Show that $\alpha f \in \mathcal{R}[a,b]$ and $\int_a^b \alpha f = \alpha \int_a^b f$.

In order to prove the extension of Theorem 3 to all of $\mathcal{R}[a,b]$, we need to show first that if $f \in \mathcal{R}[a,b]$, then $|f| \in \mathcal{R}[a,b]$. This is perhaps most easily done by first showing that the positive part of $f$ is Riemann integrable.

Given a function $f \in B[a,b]$, we define $f \vee 0$ by

$$(f \vee 0)(x) = \max(x, 0).$$

Notice that $f \le f \vee 0$, and if if $f, g \in B[a,b]$ satisfy $g \le f$, then $g \vee 0 \le f \vee 0$. It follows that

$$f + g \le f \vee 0 + g \vee 0$$

and hence

$$(f+g) \vee 0 \le (f \vee 0 + g \vee 0) \vee 0 = f \vee 0 + g \vee 0.$$

With these facts in hand, we can now show the positive part of $f$ is Riemann integrable whenver $f$ is.

**Proposition 9:** Suppose $f \in \mathcal{R}[a,b]$. Then $f \vee 0 \in \mathcal{R}[a,b]$.

*Proof.* Let $f \in \mathcal{R}[a,b]$ and let $\epsilon > 0$ Let $g$ and $G$ be step functions such that $g \le f \le G$ and such that that

$$\int_a^b (G - g) < \epsilon.$$

Now notice that $G \vee 0$ and $g \vee 0$ are step functions and

$$g \vee 0 \le f \vee 0 \le G \vee 0.$$

Moreover,

$$G \vee 0 = ((G - g) + g) \vee 0 \le (G - g) \vee 0 + g \vee 0 = (G - g) + (g \vee 0).$$

Hence

$$G \vee 0 - g \vee 0 \le G - g$$

and therefore

$$\int_a^b G \vee 0 - g \vee 0 \le \int_a^b G - g < \epsilon.$$

Hence by Proposition 5, $f \vee 0$ is Riemann integrable. □

**Corollary 10:** If $f \in \mathcal{R}[a,b]$, then $|f| \in \mathcal{R}[a,b]$ and

$$\int_a^b f \le \int_a^b |f|.$$

*Proof.* Notice that

$$f = (f \vee 0) - (-f \vee 0) \qquad \text{and} \qquad |f| = (f \vee 0) + (-f \vee 0).$$

Since $|f|$ is a sum of Riemann integrable functions, it is Riemann integrable. Moreover,

$$\int_a^b f = \int_a^b (f \vee 0) - \int_a^b (-f \vee 0) \leq \int_a^b (f \vee 0) + \int_a^b (-f \vee 0) = \int_a^b |f|.$$

□

The final property to extend is the domain decomposition of the integral.

**Proposition 11:** Let $f \in B[a,b]$ and let $c \in [a,b]$. Then $f \in \mathcal{R}[a,b]$ if and only if $f \in \mathcal{R}[a,c]$ and $f \in \mathcal{R}[c,b]$.

*Proof.* Suppose $f \in \mathcal{R}[a,b]$. Let $\epsilon > 0$ and let $G$ and $g$ be step functions such that $g \leq f \leq G$ and such that $\int_a^b (G - g) < \epsilon$. The restrictions of $G$ and $g$ to $[a,c]$ also satisfy $g \leq f \leq G$. Moreover, since $G - g \geq 0$,

$$\int_a^c (G - g) \leq \int_a^b (G - g) < \epsilon.$$

So the restriction of $f$ to $[a,c]$ is Riemann integrable, and a similar argument shows that the restriction of $f$ to $[c,b]$ is Riemann integrable.

Conversely, suppose $f \in \mathcal{R}[a,c]$ and $f \in \mathcal{R}[c,b]$. For notational convenience, let $f_1$ and $f_2$ be the restrictions of $f$ to $[a,c]$ and $[c,b]$ respectively. Let $\epsilon > 0$ and let $g_1$ and $G_1$ be step functions on $[a,c]$ such that $g_1 \leq f_1 \leq G_1$ and

$$\int_a^c (G_1 - g_1) < \epsilon/2.$$

Similarly, let $g_2$ and $G_2$ be step functions on $[c,b]$ such that $g_2 \leq f_2 \leq G_2$ and

$$\int_a^c (G_2 - g_2) < \epsilon/2.$$

Let $g$ be the step function on $[a,b]$ that is equal to $g_1$ on $[a,c)$, equal to $g_2$ on $(c,b]$ and equal to $f(c)$ at $c$. Then $g \leq f$ and

$$\int_a^b g = \int_a^c g + \int_c^b g = \int_a^c g_1 + \int_c^b g_2.$$

Let $G$ be a similarly defined step function with respect to $G_1$ and $G_2$, so $G \geq f$ and

$$\int_a^b G = \int_a^c G_1 + \int_c^b G_2.$$

Then $g \leq f \leq G$ on $[a,b]$ and

$$\int_a^b (G - g) = \int_a^c (G_1 - g_1) + \int_c^b (G_2 - g_2) < \epsilon/2 + \epsilon/2 = \epsilon.$$

Hence $f$ is Riemann integrable on $[a,b]$. □

With this last result in hand, it is not difficult to establish the extension of Theorem 4.

**Theorem 12:** Let $f \in \mathcal{R}[a,b]$. Then for any $c \in [a,b]$ we have

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

**Exercise 7:** Prove Theorem 12.