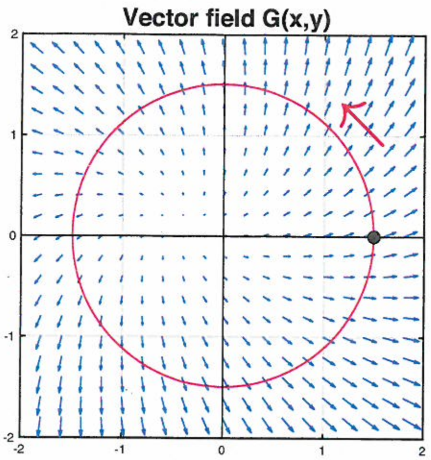
MATH 253 Name: SOLUTIONS

'Big' Quiz 12

April 19, 2022

**Instructions:** (15 points total) Show all work for credit. You may use a single formula sheet which should be handed in with your quiz.

1. (3 pts.) Consider the 2-dimensional vector field $\mathbf{G}(x,y)$ shown to the left below:




Simple closed curve C with (positive) counter-clockwise orientation shown in red

Is the vector field $\mathbf{G}(x,y)$ conservative or not? Explain briefly.

$\boxed{\text{No.}}$ $\oint_C \vec{G} \cdot d\vec{r} > 0$ since individual $\vec{G} \cdot \vec{r}\,'(t) > 0$. For a conservative vector field $\vec{G}$, this integral must be 0. Why? [ potential function $g$ $g(\text{end pt}) - g(\text{beg pt}) = 0$ OR independence of path, etc. ]

2. (5 pts.) Consider the 2-dimensional vector field

$$\mathbf{F}(x,y) = \langle 2x+y,\, x+3y \rangle .$$

Find the work done by the vector field $\mathbf{F}$ in moving a particle along the line segment from $P(1,1)$ to the $Q(2,0)$.

$\boxed{\text{Soln 1:}}$

$C =$ Line segment $\vec{PQ}$ : $\vec{r}(t) = (1-t)\langle 1,1\rangle + t\langle 2,0\rangle$ $\quad 0 \le t \le 1$

$= \langle 1+t,\, 1-t\rangle$ $\quad 0 \le t \le 1$

$$\text{Work} = \int_C \vec{F} \cdot d\vec{r} = \int_0^1 \underbrace{\langle 2(1+t) + (1-t),\ (1+t)+3(1-t)\rangle}_{\vec{F}(\vec{r}(t))} \cdot \underbrace{\langle 1,-1\rangle\, dt}_{\vec{r}\,'(t)dt}$$

$$= \int_0^1 \langle t+3,\, -2t+4\rangle \cdot \langle 1,-1\rangle\, dt$$

$$= \int_0^1 t+3+2t-4\ dt = \int_0^1 3t-1\ dt = \tfrac{3}{2}t^2 - t \Big|_0^1 = \boxed{\tfrac{1}{2}}$$

$\boxed{\text{Soln 2:}}$ Notice $\vec{F}$ is conservative with potential function $f(x,y) = x^2 + xy + \frac{3}{2}y^2$ $+c$

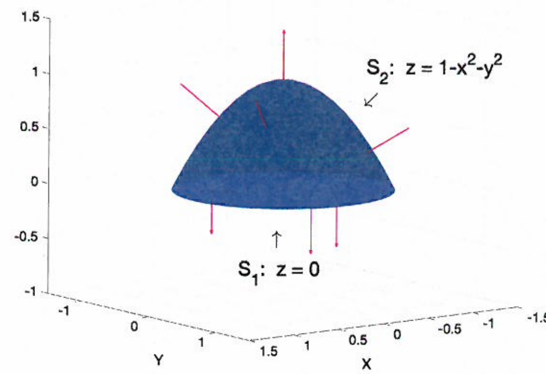
Then $\int_C \vec{F} \cdot d\vec{r} = f(2,0) - f(1,1) = (2^2) - (1 + 1 + \frac{3}{2}) = 4 - 3.5 = \boxed{\tfrac{1}{2}}$ (Easier.)

3. (7 pts.) Consider the electrical field

$$\mathbf{E}(x, y, z) = \langle y,\ x,\ z \rangle\,.$$

By Gauss' Law, the net charge enclosed by a closed surface equals the electical flux through the surface S:

Net charge enclosed by $S\ = \epsilon_0 \iint_S \mathbf{E} \cdot d\mathbf{S}.$




Find the value of the flux integral across the surface $S$ bounded by $z = 1 - x^2 - y^2$ and the $xy$-plane as directed. Let $S = S_1 \cup S_2$ as shown in the figure. Some normal vectors to the surface $S$ are shown in red.

(a) (2 pts.) Carefully and succinctly justify that

$$\epsilon_0 \iint_{S_1} \mathbf{E} \cdot d\mathbf{S} = 0$$

by considering the surface $S_1$ (disk in $xy$-plane defined by $z = 0$) and the electrical field $\mathbf{E}$.

*Answer: The flux integral through $S_1$ is zero because .....*

On $S_1$, $\vec{E} = \langle x, y, 0\rangle$ and a normal vector is $\vec{n} = \langle 0, 0, -1\rangle = -\hat{k}$.

Thus, $\vec{E} \cdot \vec{n}\, dS = \langle x, y, 0\rangle \cdot \langle 0, 0, -1\rangle = 0$. The integrand is 0.

Concise version: "Since $\vec{E} = \langle x, y, 0\rangle$ on $S_1$, there is no flow through $S_1$."

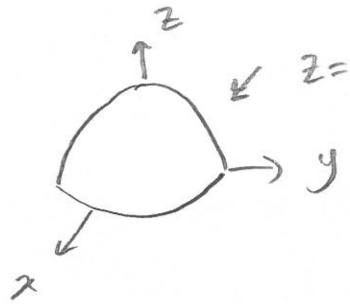
(b) (5 pts.) From (a) and Gauss' Law, you now know that the net charge enclosed by $S$ is

$$\epsilon_0 \iint_{S_2} \mathbf{E} \cdot d\mathbf{S}.$$

Compute this flux integral. (Next page is blank for additional work.)

$\epsilon_0 \iint_S \vec{E} \cdot d\vec{S} = \epsilon_0 \iint_{S_2} \vec{E} \cdot d\vec{S}$ since $\epsilon_0 \iint_{S_1} \vec{E} \cdot d\vec{S} = 0$ by (a)

$\rightarrow$




$z = 1 - x^2 - y^2$ $\quad S_2$ $\qquad$ Graph!

$\vec{E} = \langle y, x, z \rangle$

$$\epsilon_0 \iint_{S_2} \vec{E} \cdot d\vec{S} = \epsilon_0 \iint_{S_2} -P f_x - Q f_y + R \; dA$$

$= 1 - x^2 - y^2$ on $S_2$

$$= \epsilon_0 \iint_{S_2} -y(-2x) - x(-2y) + z \quad dA$$

$$= \epsilon_0 \iint_{S_2} 4xy + 1 - x^2 - y^2 \; dA$$

$$= \epsilon_0 \int_0^{2\pi} \int_0^1 (4 r\cos\theta (r\sin\theta) + 1 - r^2) \, r \, dr \, d\theta$$

$$= \epsilon_0 \int_0^{2\pi} \int_0^1 4r^3 \cos\theta \sin\theta + r - r^3 \; dr \, d\theta$$

$$= \epsilon_0 \int_0^{2\pi} \cos\theta \sin\theta \, r^4 + \tfrac{1}{2} r^2 - \tfrac{1}{4} r^4 \Big|_0^1 \; d\theta$$

$$= \epsilon_0 \int_0^{2\pi} \cos\theta \sin\theta + \tfrac{1}{4} \; d\theta = \epsilon_0 \left[ \tfrac{1}{2} \sin^2\theta + \tfrac{\theta}{4} \right]_0^{2\pi}$$

$$= \epsilon_0 \left[ \left( \tfrac{1}{2}(0) + \tfrac{2\pi}{4} \right) - 0 \right] = \epsilon_0 \frac{\pi}{2}$$

Aside: $d\vec{S} = \langle -f_x, -f_y, 1 \rangle = \langle 2x, 2y, 1 \rangle$

points "outward" or "upward"