MATH 253 Name: SOLUTIONS

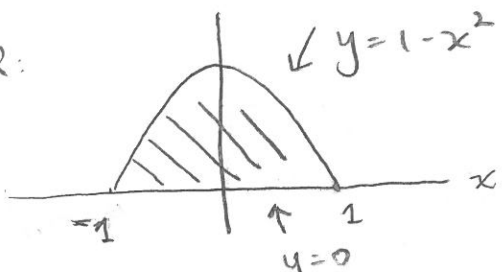
Quiz 9 March 25, 2022

Instructions: Five points total. Show all work for credit.

1. (5 pts.) Find the mass m and center of mass $(\bar{x}, \bar{y})$ of the lamina R that is bounded by $y = 1 - x^2$ and the x-axis, when the density function is given by $\rho(x, y) = 10y\ \ mg/cm^2$

   (a) (2 pts.) Compute the mass m of the lamina, including units in your answer.

R:



$$m = \iint_R \rho(x,y)\, dA$$

$$= \int_{-1}^{1} \int_0^{1-x^2} 10y \, dy\, dx$$

$$= \int_{-1}^{1} 5y^2 \Big|_{y=0}^{y=1-x^2} dx = \int_{-1}^{1} 5(1-x^2)^2\, dx$$

$$= 5 \int_{-1}^{1} 1 - 2x^2 + x^4 \, dx = 5\left[x - \frac{2}{3}x^3 + \frac{1}{5}x^5\right]\Big|_{-1}^{1}$$

$$= 5\left[\left(1 - \frac{2}{3} + \frac{1}{5}\right) - \left(-1 + \frac{2}{3} - \frac{1}{5}\right)\right] = 10\left(\frac{8}{15}\right) = \boxed{\frac{16}{3}\ mg}$$

   (b) (1 pt) You could do a complicated integral to conclude that the moment about the y-axis, $M_y = 0$. Instead, give a reason that $M_y = 0$ **without using Calculus at all** by thinking. If you are at a loss initially, think about the density function $\rho(x, y)$, the lamina, and the formula for M_y.

Answer: $M_y = 0$ because .... M_y measures the tendency to rotate about the y-axis and, (you want to notice 1)) the density $\rho(x,y)$ depends only on y and 2) the lamina is symmetric about the y-axis. Putting this all together the balance point is $\bar{x} = 0$.

(c) (2 pts.) Now compute the center of mass $(\bar{x}, \bar{y})$, showing all work for credit. What are the units?

$\bar{x} = 0$ from part (b)

For $\bar{y}$: $M_x = \iint_R y\, \rho(x,y)\, dA = \int_{-1}^{1} \int_0^{1-x^2} y(10y)\, dy\, dx$

$$= \int_{-1}^{1} \int_0^{1-x^2} 10y^2\, dy\, dx = \int_{-1}^{1} \frac{10}{3} y^3 \Big|_0^{1-x^2} dx$$

$$= \int_{-1}^{1} \frac{10}{3}(1-x^2)^3\, dx = \frac{10}{3} \int_{-1}^{1} 1 - 3x^2 + 3x^4 - x^6\, dx$$

$$= \frac{10}{3}\left[x - x^3 + \frac{3}{5}x^5 - \frac{1}{7}x^7\right]_{-1}^{1} = \frac{10}{3}\left[\left(1-1+\frac{3}{5}-\frac{1}{7}\right) - \left(-1+1-\frac{3}{5}+\frac{1}{7}\right)\right]$$

$$= \frac{10}{3} \cdot 2\left(\frac{3}{5} - \frac{1}{7}\right) = \frac{20}{3}\left[\frac{21-5}{35}\right] = \frac{20}{3} \cdot \frac{16}{35} = \frac{4 \cdot 16}{3 \cdot 7} = \frac{64}{21}$$

$$\bar{y} = \frac{M_x}{m} = \frac{\frac{64}{21}\ (\text{cm})(\text{mg})}{\frac{16}{3}\ \text{mg}} = \boxed{\frac{4}{7}\ \text{cm}}$$

Units:

$$\underset{y}{\text{cm}}\ \underset{\rho(x,y)}{\left(\frac{\text{mg}}{\text{cm}^2}\right)}\ \underset{dA}{\text{cm}^2} = (\text{cm})(\text{mg})$$

$$\boxed{(\bar{x}, \bar{y}) = \left(0, \frac{4}{7}\right)\ \text{cm}}$$