MATH 253 Name: SOLUTIONS

Quiz 7 February 25, 2022

Instructions: Ten points total. Show all work for credit.

1. (5 pts.) Consider the implicitly defined surface given by equation

$$e^x = 5xyz$$

(a) Find a point $P = P\left(1, \frac{e}{10}, c\right)$ on the surface with x-coordinate equal to 1, and y-coordinate equal to $\frac{e}{10}$. (This means find c.)

$x=1 \quad y=\frac{e}{10} \Rightarrow e^1 = 5(1)\frac{e}{10}z \Rightarrow z=2$

Answer: The coordinates of P are $\boxed{\left(1, \frac{e}{10}, 2\right)}$ $\boxed{z=2}$

(b) Using the point P found in the last part, find the equation of the tangent plane to the surface at $\left(1, \frac{e}{10}, c\right)$.

Since the surface is defined implicity, a normal vector is

$\vec{n} = \nabla f(1, \frac{e}{10}, 2)$ where $f(x,y,z) = e^x - 5xyz$

Thus, $\nabla f = \langle f_x, f_y, f_z \rangle = \langle e^x - 5yz, -5xz, -5xy \rangle$ and

$\nabla f(1, \frac{e}{10}, 2) = \langle e - 5(\frac{e}{10})(2), -5(1)(2), -5(1)(\frac{e}{10}) \rangle$

$= \langle 0, -10, \frac{-e}{2} \rangle$

I'll use $\vec{n} = -\nabla f(1, e/10, 2) = \langle 0, 10, \frac{e}{2} \rangle$ for the normal vector.

Equation of tangent plane is:

$\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{p}$ or $\langle 0, 10, \frac{e}{2} \rangle \cdot \langle x, y, z \rangle = \langle 0, 10, \frac{e}{2} \rangle \cdot \langle 1, \frac{e}{10}, 2 \rangle$

$10y + \frac{e}{2}z = e + e$

$\boxed{10y + \frac{e}{2}z = 2e}$ or, $20y + ez = 4e$ etc.

(continued on next page)

(c) Find the partial derivatives $\frac{\partial x}{\partial y}$ and $\frac{\partial z}{\partial y}$ for the surface $\boxed{e^x = 5xyz.}$

Need the Chain Rule for implicitly-defined functions

$\frac{\partial x}{\partial y}$:

x dependent

$$e^x = 5xyz$$

product rule

$$e^x \frac{\partial x}{\partial y} = 5z\left[x + \frac{\partial x}{\partial y} y\right]$$

$$e^x \frac{\partial x}{\partial y} = 5xz + 5yz \frac{\partial x}{\partial y}$$

$$\frac{\partial x}{\partial y}(e^x - 5yz) = 5xz \Rightarrow \boxed{\frac{\partial x}{\partial y} = \frac{5xz}{e^x - 5yz}}$$

$\frac{\partial z}{\partial y}$:

$$e^x = 5xyz$$

z dependent on (x,y)

Soln 1: Implicit diff.

$$0 = 5x\left[y\frac{\partial z}{\partial y} + (1)z\right]$$

$$\Rightarrow \frac{\partial z}{\partial y}(5xy) = -5xz$$

$$\Rightarrow \frac{\partial z}{\partial y} = \frac{-5xz}{5xy}$$

$$\Rightarrow \boxed{\frac{\partial z}{\partial y} = \frac{-z}{y} \quad x \neq 0, \; y \neq 0}$$

Soln 2: Solve for z first.

$$z = \frac{e^x}{5x} y^{-1}$$

so that

$$\frac{\partial z}{\partial y} = \frac{e^x}{5x}(-1)y^{-2} = \frac{-e^x}{5xy^2}$$

$$\boxed{\frac{\partial z}{\partial y} = \frac{-e^x}{5xy^2}}$$

Answers: $\frac{\partial x}{\partial y} = \frac{5xz}{e^x - 5yz}$

$$\boxed{\frac{\partial z}{\partial y} = \frac{-z}{y} \text{ or } \frac{-e^x}{5xy^2}}$$

These are the same! Why?

2. (3 pts.) Find the directional derivative of the function $f(x,y) = e^y \sin(x)$ at the point $(\frac{\pi}{3}, 0)$ in the direction of $\mathbf{v} = \langle 8, -6 \rangle$.

- unit vector $\vec{u}$ in direction of $\vec{v}$ is $\langle \frac{4}{5}, \frac{-3}{5} \rangle$ [Use 3-4-5 triple.]

- $\nabla f(x,y) = \langle f_x, f_y \rangle = \langle e^y \cos x, e^y \sin(x) \rangle$

  $\nabla f(\frac{\pi}{3}, 0) = \langle e^0 \cos(\frac{\pi}{3}), e^0 \sin(\frac{\pi}{3}) \rangle = \langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle$

- $D_{\vec{u}} f(\frac{\pi}{3}, 0)) = \nabla f(\frac{\pi}{3}, 0) \cdot \vec{u} = \langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle \cdot \langle \frac{4}{5}, \frac{-3}{5} \rangle = \frac{4}{10} - \frac{3\sqrt{3}}{10}$

  $= \boxed{\frac{4 - 3\sqrt{3}}{10}}$

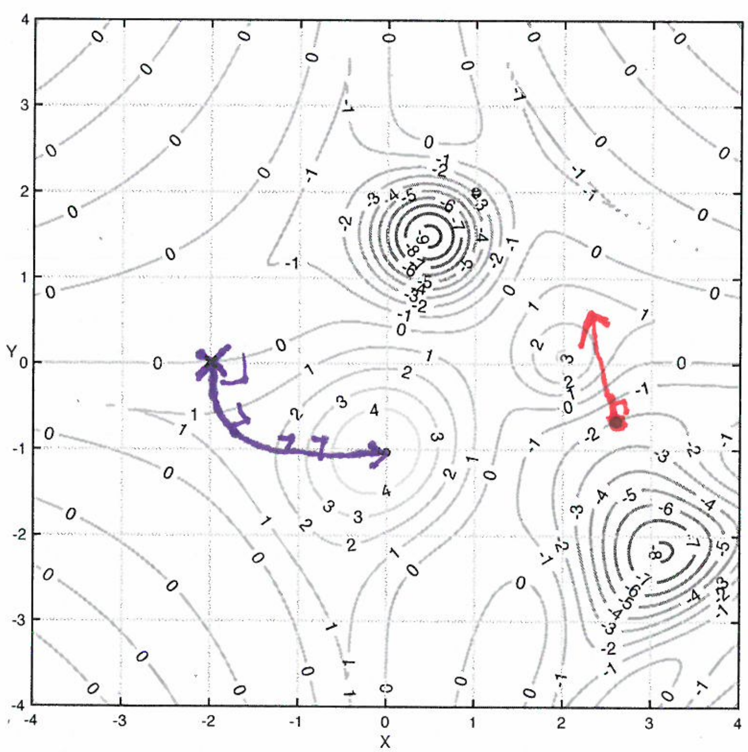
Is $f(x,y)$ increasing / decreasing / stable at $(\frac{\pi}{3}, 0)$ in the direction of $\mathbf{v}$? Explain.

$\frac{4-3\sqrt{3}}{10} < 0$

3. (2 pts.) Consider the contour plot for the smooth function $z = f(x,y)$ displayed below.

(b) orthogonal to level curves and seeks positive charge.

(*) -1 pt if $\underline{\underline{\text{not}}}$ orthogonal to level curves



$\nabla f(2.6, -0.7)$ should be 1) orthogonal to level curve and 2) point in the direction of increase.

(a) At the red point $(2.6, -0.7)$ shown, draw a vector pointing in the direction of $\nabla f(2.6, -0.7)$.

(b) Suppose a negatively charged particle is placed at the black X at $(-2, 0)$, and that $f(x,y)$ gives the charge of a plate in coulombs. Sketch the path of the negatively charged particle on the plate.