MATH 253
Quiz 5

Name: SOLUTIONS

February 11, 2022

Instructions: Ten points total. Show all work for credit.

1. (4 pts.)

   (a) (2 pts.) Prove that the following limit does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^6+3y^2}$$

- Along $x=0$, $(x,y)=(0,y)$ so

$$\lim_{(0,y)\to(0,0)} \frac{x^3y}{x^6+3y^2} = \lim_{y\to 0} \frac{0^3y}{0^6+3y^2} = \lim_{y\to 0} \frac{0}{3y^2} = \boxed{0}$$

- Along $y=x^3$, $(x,y)=(x,x^3)$ so

$$\lim_{(x,x^3)\to(0,0)} \frac{x^3y}{x^6+3y^2} = \lim_{x\to 0} \frac{x^3(x^3)}{x^6+3(x^3)^2} = \lim_{x\to 0} \frac{x^6}{4x^6} = \boxed{\frac{1}{4}}$$

Since $0 \neq \frac{1}{4}$, this limit d.n.e.

   (b) (2 pts.) Find the value of the limit below and give a brief mathematical justification that this limit exists.

$$\lim_{(x,y)\to(3,2)} \frac{xy}{\sin\left(\frac{\pi}{y}\right)+e^{3y-2x}}$$

The function $f(x,y) = \dfrac{xy}{\sin(\pi/y)+e^{2y-3x}}$ is continuous at $(3,2)$!

Since the denominator $\sin(\pi/2)+e^{3(2)-2(3)} = 1+1 = 2 \neq 0$.

Thus $$\lim_{(x,y)\to(3,2)} \frac{xy}{\sin(\frac{\pi}{y})+e^{3y-2x}} = \frac{3(2)}{2} = \boxed{3}$$

2. (6 pts.) Consider the function $g(x, y) = \sin\left(\frac{y}{1+x}\right)$.

(a) (2 pt.) Is the function $g(x, y)$ increasing, decreasing, or stable in the x-direction at the point in its domain $P(2, \pi,)$? Briefly justify your answer.

Translation: Compute $\frac{\partial g}{\partial x}$ at $(2,\pi)$ and interpret its sign.

$$\frac{\partial g}{\partial x} = \cos\left(\frac{y}{1+x}\right)\left[y(-1)(1+x)^{-2}\right] = \frac{-y\cos\left(\frac{y}{1+x}\right)}{(1+x)^2}$$

$$\frac{\partial g}{\partial x}(2,\pi) = \frac{-\pi\cos\left(\frac{\pi}{1+2}\right)}{(1+2)^2} = \frac{-\pi\cos\left(\frac{\pi}{3}\right)}{3^2} = \frac{-\pi\frac{1}{2}}{9} = \boxed{\frac{-\pi}{18}}$$

Since $-\pi/18 < 0$, $g(x,y)$ is $\boxed{\text{decreasing}}$ with respect to x at $(2,\pi)$.

(b) (4 pts.) Find the equation of the tangent plane to $g(x, y)$ at the point $(2, \pi, g(2, \pi))$.

$$\Delta z = \frac{\partial g}{\partial x}(2,\pi)\,\Delta x + \frac{\partial g}{\partial y}(2,\pi)\,\Delta y$$

$$z - \frac{\sqrt{3}}{2} = \frac{-\pi}{18}(x-2) + \frac{1}{6}(y-\pi)$$

$$z = -\frac{\pi}{18}x + \frac{\pi}{9} + \frac{1}{6}y - \frac{\pi}{6} + \frac{\sqrt{3}}{2}$$

$$\boxed{z = \frac{-\pi}{18}x + \frac{1}{6}y - \frac{\pi}{18} + \frac{\sqrt{3}}{2}}$$

Computations to right.

$$g(2,\pi) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

Value at $(2,\pi)$

$$\frac{\partial g}{\partial y} = \cos\left(\frac{y}{1+x}\right)\cdot\frac{1}{1+x} = \frac{\cos\left(\frac{y}{1+x}\right)}{1+x}$$

$$\left.\frac{\partial g}{\partial y}\right|_{(2,\pi)} = \frac{\cos\left(\frac{\pi}{3}\right)}{3} = \frac{1}{6}$$

$$\frac{\pi}{9} - \frac{\pi}{6} = \frac{-\pi}{18}$$