MATH 253
Quiz 3

Name: SOLUTIONS
Due at 11:59 pm in GS on Monday, January 31.

Instructions: (10 pts.) Show all work for credit. You may use your book, but no other resource. Bald answers will be given zero credit.

1. Suppose a particle moves in 3-space and you record its trajectory for times $t \in \left[-\frac{\pi}{8}, \frac{\pi}{8}\right]$. This is given by the space curve with equation

$$\mathbf{r}(t) = \langle \sin(2t), \ln(\cos(2t)), \cos(2t) \rangle \qquad \text{for } t \in \left[-\frac{\pi}{8}, \frac{\pi}{8}\right]$$

where t is measured in seconds and the coordinate functions $x(t)$, $y(t)$, and $z(t)$ in meters.

(a) (2pts.) Give the coordinates of the particle in $\mathbb{R}^3$ at times $t = -\frac{\pi}{8}, 0, \frac{\pi}{8}$. After giving an exact value, round your answer to two decimal places ~~if appropriate~~. to get an estimate for the logarithm.

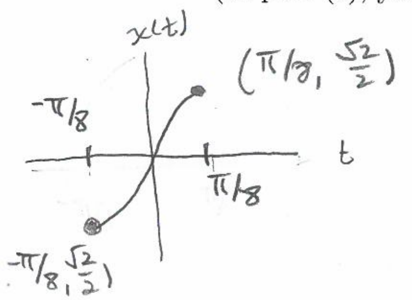
$\vec{r}(-\pi/8) = \langle \sin(2(-\pi/8)), \ln(\cos(2(-\pi/8))), \cos(2(-\pi/8)) \rangle = \langle \sin(-\pi/4), \ln(\cos(-\pi/4)), \cos(-\pi/4) \rangle$

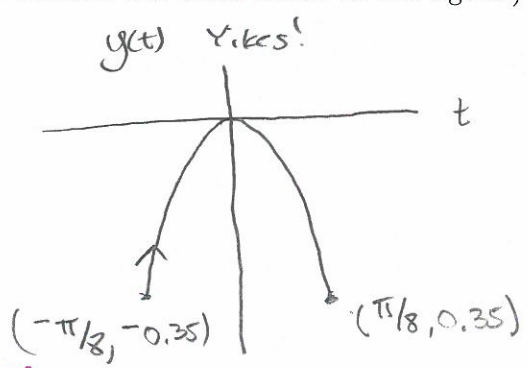
$= \langle -\frac{\sqrt{2}}{2}, \ln(\frac{\sqrt{2}}{2}), \frac{\sqrt{2}}{2} \rangle \approx \boxed{\langle -0.71, -0.35, 0.71 \rangle \text{ or } \langle -\frac{\sqrt{2}}{2}, -0.35, \frac{\sqrt{2}}{2} \rangle}$

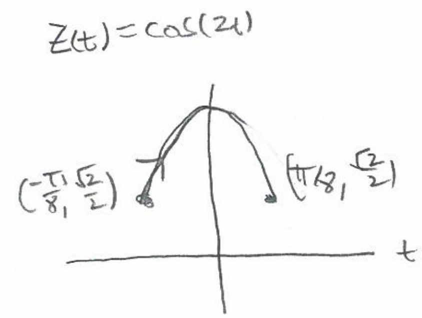
$\vec{r}(0) = \langle \sin(0), \ln(\cos(0)), \cos(0) \rangle = \boxed{\langle 0, 0, 1 \rangle}$

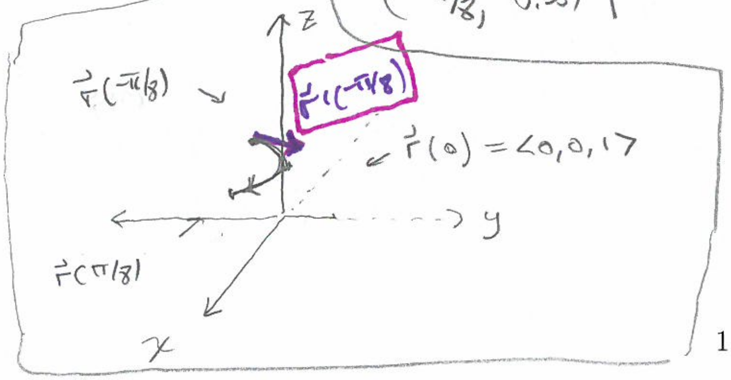
$\vec{r}(\pi/8) = \langle \sin(\pi/4), \ln(\cos(\pi/4)), \cos(\pi/4) \rangle = \boxed{\langle \frac{\sqrt{2}}{2}, \ln(\frac{\sqrt{2}}{2}), \frac{\sqrt{2}}{2} \rangle \approx \langle 0.71, -0.35, 0.71 \rangle}$

(b) (2 pts.) By thinking about the coordinate functions, sketch the trajectory of the particle over the time period $-\frac{\pi}{8} \le t \le \frac{\pi}{8}$ seconds. Label the three points from part (a) on the trajectory, and put arrows on the path to display the direction of travel. [FYI: The trajectory is not that interesting.] (In part (c), you will add one more vector to this figure.)









You don't need these component plots, but they might be helpful.

(c) (3 pts.) Find the velocity vector $\mathbf{r}'(t)$ and the speed of the particle at time $t = -\frac{\pi}{8}$. Include units in your answer. Finally, returning to (b), draw the velocity vector $\mathbf{r}'(-\frac{\pi}{8})$ with its base (beginning point) at the position of the particle $\mathbf{r}(-\frac{\pi}{8})$.

in purple.

$\vec{r}(t) = \langle \sin(2t), \ln(\cos(2t)), \cos(2t) \rangle$

$\vec{r}'(t) = \langle \cos(2t)\cdot 2, \frac{1}{\cos(2t)}\cdot -\sin(2t)\cdot 2, -\sin(2t)\cdot 2 \rangle$

$= \langle 2\cos(2t), -2\tan(2t), -2\sin(2t) \rangle$

$\vec{r}'(-\pi/8) = \langle 2\cos(2(-\frac{\pi}{8})), -2\tan(2(-\pi/8)), -2\sin(2(-\pi/8)) \rangle$

$= \langle 2\cos(-\pi/4), 2\tan(-\pi/4), -2\sin(-\pi/4) \rangle = \boxed{\langle \sqrt{2}, 2, \sqrt{2} \rangle}$

Speed = magnitude of velocity vector

$|\vec{r}'(\pi/8)| = \sqrt{(\sqrt{2})^2 + (2)^2 + (\sqrt{2})^2} = \boxed{\sqrt{8} = 2\sqrt{2} \text{ meters/second}}$

(d) (3 pts.) Find the distance that the particle travels between time $t = -\frac{\pi}{8}$ and $t = \frac{\pi}{8}$ seconds. Include units.

$L = \text{arclength} = \int_{-\pi/8}^{\pi/8} |\vec{r}'(t)|\,dt = \int_{-\pi/8}^{\pi/8} \sqrt{(2\cos(2t))^2 + (-2\tan(2t))^2 + (-2\sin(2t))^2}\,dt$

$= \int_{-\pi/8}^{\pi/8} \sqrt{4\cos^2(2t) + 4\sin^2(2t) + 4\tan^2(2t)}\,dt = 2\int_{-\pi/8}^{\pi/8} \sqrt{1+\tan^2(2t)}\,dt$

$= 2\int_{-\pi/8}^{\pi/8} \sec 2t\,dt =$ Let $u = 2t$ $du = 2dt$ $dt = \frac{1}{2}du$

upper limit: $\pi/4$
lower limit: $-\pi/4$

$= \int_{-\pi/4}^{\pi/4} \sec(u)\,du = \ln|\sec(u)+\tan(u)|\Big|_{-\pi/4}^{\pi/4} = \ln|\sec(\pi/4)+\tan(\pi/4)| - \ln|\sec(-\pi/4)+\tan(-\pi/4)|$

new limits

$= \ln(\sqrt{2}+1) - \ln|\sqrt{2}-1| \approx \boxed{1.76 \text{ meters}}$ $= \ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$