Name: Solutions

1. The following vector field is conservative:

$$\mathbf{F} = \langle ye^z, xe^z - z, xye^z - y \rangle$$

a) Find a potential function for **F**.

$$\frac{\partial f}{\partial x} = ye^z \Rightarrow f(x,y,z) = xye^z + h(y,z)$$

$$\frac{\partial f}{\partial y} = xe^z - z \Rightarrow xe^z + \frac{\partial h}{\partial y} = xe^z - z$$

$$\Rightarrow \frac{\partial h}{\partial y} = -z$$

$$\Rightarrow h = -zy + g(z)$$

$$\frac{\partial f}{\partial z} = xye^z - y \Rightarrow xye^z - y + g'(z) = xye^z - y$$

$$\Rightarrow g'(z) = 0 \Rightarrow g = C \quad (C = 0 \text{ e.g.})$$

$$\boxed{f(x,y,z) = xye^z - yz}$$

b) Doing very little work, compute $\int_C \mathbf{F} \cdot d\mathbf{R}$ where C is the straight line from the point $\langle 1, 1, 0 \rangle$ to the point $\langle 0, 1, 2 \rangle$.

Fund. Thm. of Line Integrals:

$$\int_C \vec{F} \cdot d\vec{r} = f(0,1,2) - f(1,1,0)$$

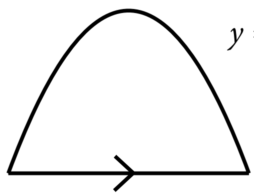
$$= (0 - 1\cdot 2) - (1 \cdot 1 \cdot e^0 - 1 \cdot 0)$$

$$= \boxed{-3}$$

2. Recall that Green's Theorem states that for any curve C traversing the boundary (counterclockwise) of a simply connected region $\mathcal{D}$

$$\int_C P\,dx + Q\,dy = \iint_{\mathcal{D}} \left(-\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}\right) dA.$$

Use Green's theorem to compute the line integral $\int_C xy\,dx + (x-y)\,dy$ where C is the boundary of the region lying between the line $y = 0$ and the graph of $y = 4 - x^2$, oriented counterclockwise. For full credit, your solution must employ Green's Theorem.



$P = xy, \quad Q = x - y$

$P_y = x, \quad Q_x = 1$

$$\iint_{\mathcal{D}} -P_y + Q_x\,dA = \int_{-2}^{2}\int_0^{4-x^2} (-x+1)\,dy\,dx$$

$$= \int_{-2}^{2} (-x+1)(4-x^2)\,dx$$

$$= \int_{-2}^{2} -4x + x^3 + 4 - x^2\,dx$$

$$= \int_{-2}^{2} 4 - x^2\,dx \quad \text{(by symmetry)}$$

$$= 4x - \frac{x^3}{3}\Big|_{-2}^{2} = \left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right)$$

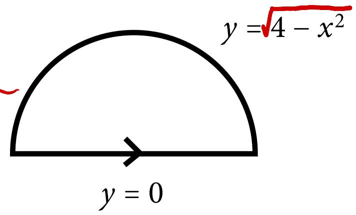
$$= 2\left(8 - \frac{8}{3}\right) = \boxed{\frac{32}{3}}$$

2. Recall that Green's Theorem states that for any curve C traversing the boundary (counterclockwise) of a simply connected region $\mathcal{D}$

Assuming $y = \sqrt{4-x^2}$ as hinted by diagram.

$$\int_C P\,dx + Q\,dy = \iint_{\mathcal{D}} \left(-\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x} \right) dA.$$

Use Green's theorem to compute the line integral $\int_C xy\,dx + (x-y)\,dy$ where C is the boundary of the region lying between the line $y = 0$ and the graph of $y = \sqrt{4-x^2}$, oriented counterclockwise. For full credit, your solution must employ Green's Theorem.



$-P_y = -x, \quad Q_x = 1$

$$\iint_{\mathcal{D}} 1 - x\,dA = \int_0^{\pi}\int_0^2 (1 - r\cos\theta)\,r\,dr\,d\theta$$

$$= \int_0^{\pi}\int_0^2 (r - r^2\cos\theta)\,dr\,d\theta$$

$$= \int_0^{\pi} \frac{r^2}{2} - \frac{r^3}{3}\cos\theta \Big|_0^2 d\theta$$

$$= \int_0^{\pi} 2 - \frac{8}{3}\cos\theta\,d\theta$$

$$= 2\pi - \frac{8}{3}\cos\theta \Big|_0^{\pi}$$

$$= \boxed{2\pi}$$