Quiz 12
Math 253

Name: Key

11/26

Stokes' Theorem states the following equation:

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{r}.$$

For the vector field $\mathbf{F}(x, y, z) = \langle x, xy, z \rangle$ and the surface S that is the part of the paraboloid $z = 5 - x^2 - y^2$ with $z \geq 4$, oriented upward, fully set up, but **do not evaluate**, integrals to compute both sides of the equation. Your answers should be in a form where only basic 1- and 2- dimensional integrals remain to be evaluated

1. The line integral:

$\vec{r}(t) = \langle \cos t, \sin t, 4 \rangle \quad t \in [0, 2\pi]$

$d\vec{r} = \langle -\sin t, \cos t, 0 \rangle$

$$\oint_{\partial S} \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \langle \cos t, \cos t \sin t, 4 \rangle \cdot \langle -\sin t, \cos t, 0 \rangle \, dt$$

$$= \int_0^{2\pi} \left(-\cos t \sin t + \cos^2 t \sin t \right) dt$$

2. The surface integral:

$\vec{r}(u, v) = \langle u, v, 5 - u^2 - v^2 \rangle$

$(u, v) \in D$ the unit disc at the origin

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & xy & z \end{vmatrix} = \langle 0, 0, y \rangle$$

$\vec{r}_u = \langle 1, 0, -2u \rangle$

$\vec{r}_v = \langle 0, 1, -2v \rangle$

$\vec{r}_u \times \vec{r}_v = \langle 2u, 2v, 1 \rangle$ Note upward normal direction

$$\iint_S \nabla \times F \cdot dS = \iint_D \langle 0, 0, v \rangle \cdot \langle 2u, 2v, 1 \rangle \, dA = \iint_D v \, dA$$

$$= \int_0^{2\pi} \int_0^1 r \sin\theta \, r \, dr \, d\theta \quad \text{or} \quad \int_{-1}^{1} \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} v \, dv \, du$$