MATH 253
Midterm 2

Name: SOLUTIONS

March 22, 2022

Instructions: 100 points total. Use only your brain, a writing implement, and a single formula sheet. Your formula sheet, with your name on it, must be turned in with your exam. You have 90 minutes to complete this exam which is six pages in length. Answers should be given in 'good' mathematical form (simplified, etc.) If you can not do a problem, move on. Good luck.

1. (20 pts.) Consider the surface defined by the graph of the function $f(x,y) = e^{3x}\cos(2y)$.

   (a) (11 pts.) Consider the point $(0,0)$ in the domain of $f(x,y)$.

   i. (7 pts.) Find the equation of the tangent plane to the surface at $(0,0,f(0,0))$.

$z = f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0)$

$= 1 + 3x$

$f(0,0) = e^0\cos(0) = 1$

$f_x = 3e^{3x}\cos(2y)$

$f_x(0,0) = 3e^0\cos(0) = 3$

$f_y = -2e^{3x}\sin(2y)$

$f_y(0,0) = -2e^0\sin(0) = 0$

$\boxed{z = 3x+1}$

   ii. (4 pts.) Estimate the value of $f(-0.01, 0.02)$ using the best linear approximation for $f(x,y)$ at $(0,0)$.

Use the tangent plane equation: $f(-0.01, 0.02) \approx 1 + 3(-0.01) = \boxed{.97}$

   (b) (9 pts.) Now consider the point $P(a,b) = (-1, \frac{\pi}{8})$ in the domain of $f(x,y)$.

   i. (6 pts.) What is the rate of change of $f(x,y)$, if you move from P in the direction south-west?

$D_{\vec{u}}(-1, \frac{\pi}{8}) = \nabla f(-1, \frac{\pi}{8}) \cdot \vec{u}$ where $\vec{u} = \langle \frac{-\sqrt{2}}{2}, \frac{-\sqrt{2}}{2} \rangle$ unit vector, SW

$\nabla f(-1, \frac{\pi}{8}) = \langle 3e^{3(-1)}\cos(\frac{2\pi}{8}), -2e^{3(-1)}\sin(\frac{2\pi}{8}) \rangle = \langle 3e^{-3}(\frac{\sqrt{2}}{2}), -2e^{-3}\frac{\sqrt{2}}{2} \rangle$

$= \frac{1}{e^3} \langle \frac{3\sqrt{2}}{2}, -\sqrt{2} \rangle$

$\nabla f(-1, \pi/8) \cdot \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle = \frac{1}{e^3} \langle \frac{3\sqrt{2}}{2}, -\sqrt{2} \rangle \cdot \langle \frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \rangle = \frac{1}{e^3}(-\frac{3}{2} + 1) = \boxed{\frac{-1}{2e^3}}$

(Problem continued on next page.)

ii. (3 pts.) (Circle one) Is $f(x,y)$ increasing / decreasing / stable as you move from P in the south-west direction? Explain briefly.

$D_{\hat{u}} f(-1, \pi/8) < 0$

2. (15 pts.) Over spring break you snow shoe up Donnelly dome. Its height is given by the function

$$h(x,y) = 4 + e^{-x^2-3y^2} \quad \text{thousand feet,}$$

where (x,y) give the grid locations on your map and are measured in kilometers.

(a) (7 pts.) You start your hike at $(x,y) = (-1,1)$ and want to ascend Donnelly dome by the steepest route possible. In what direction **u** should you move from $(-1,1)$ to accomplish this? Give your final answer **u** as a **unit vector**.

$\vec{u} = \dfrac{\nabla h(-1,1)}{|\nabla h(1,1)|}$

$\nabla h \ \langle -2xe^{-x^2-3y^2}, -6ye^{-x^2-3y^2} \rangle$

$\nabla h(-1,1) = \langle -2(-1)e^{-(-1)^2-3(1)^2}, -6(1)e^{-4} \rangle = 2e^{-4} \underbrace{\langle 1, -3 \rangle}_{\text{direction!}}$

Thus, $\vec{u} = \dfrac{\langle 1,-3 \rangle}{|\langle 1,-3 \rangle|} = \dfrac{1}{\sqrt{10}} \langle 1,-3 \rangle$

Answer: **u** = $\langle \frac{1}{\sqrt{10}}, \frac{-3}{\sqrt{10}} \rangle$

(b) (4 pts.) At what rate are you ascending Donnelly dome in this (steepest ascent) direction? Include units in your answer.

$|\nabla h(-1,1)| = |2e^{-4} \langle 1,-3 \rangle| = 2e^{-4} |\langle -1,3 \rangle| = 2\sqrt{10}\, e^{-4}$

$\boxed{\dfrac{2\sqrt{10}}{e^4} \quad \text{thousand ft per Km}}$

units

(c) (4 pts.) Without doing any (significant) computation at all, but by thinking about the height function $h(x,y)$, find the maximum height of Donnelly dome, and the grid coordinates (a,b) where that maximum height occurs.

Answer: Max occurs at $(0,0)$ $\quad h(0,0) = 4+1 = 5$ thousand feet

$\boxed{5000 \text{ ft}}$

3. (15 pts.) Let $f(x, y) = 2 - x^4 + 2x^2 - y^2$.

(a) (8pts.) Find all three critical points of $f(x, y)$.

Simultaneously solve $f_x = 0$, $f_y = 0$ for critical points.

$f_x = -4x^3 + 4x = 0$ $\quad$ $f_y = -2y = 0$

(1) $-4x(x^2-1) = 0$ $\quad$ (2) $-2y = 0$

From (1), $x = 0, \pm 1$, from (2), $y = 0$.

The critical points are $\boxed{(0,0),\ (1,0),\ (-1,0)}$

(b) $f_{xx} = -12x^2 + 4$ $\quad$ $f_{yy} = -2$ $\quad$ $f_{xy} = 0$

Thus, $D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} -12x^2 + 4 & 0 \\ 0 & -2 \end{vmatrix} = 24x^2 - 8$

(b) (7 pts.) Use the second derivatives test to classify these as local maxima, local minima, saddle points, or there is not enough information to tell.

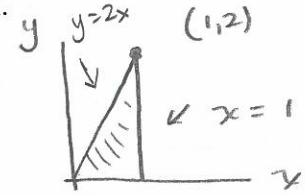
critical point	$D = 24x^2 - 8$	f_{xx}	Conclusion
$(0,0)$	$D = -8 < 0$		saddle point
$(1,0)$	$D = 16 > 0$	$f_{xx}(1,0) = -8 < 0$	local max
$(-1,0)$	$D = 16$	$f_{xx}(-1,0) = -8 < 0$	local max

4. (10 pts.) Consider the iterated integral: $$\int_0^2 \int_{\frac{y}{2}}^1 y\cos(x^3-1)\,dx\,dy$$

(a) (2 pts.) Sketch the region of integration for the iterated integral.

$\frac{y}{2} \le x \le 1$

$0 \le y \le 2$



If $x = \frac{y}{2}$, then $y = 2x$

(b) (8 pts.) Compute the value of the integral by interchanging the order of integration.

$$\int_0^2 \int_{\frac{y}{2}}^1 y\cos(x^3-1)\,dx\,dy = \int_0^1 \int_0^{2x} y\cos(x^3-1)\,dy\,dx$$

$$= \int_0^1 \cos(x^3-1)\left[\tfrac{1}{2}y^2\right]_0^{2x} dx = \int_0^1 2x^2\cos(x^3-1) - 0 \; dx$$

$$= \int_0^1 2x^2\cos(x^3-1)\,dx$$

If $u = x^3-1$, then $du = 3x^2\,dx$ and $2x^2\,dx = \frac{2}{3}du$

$$= \int_{u=-1}^{u=0} \frac{2}{3}\cos(u)\,du$$

$x=0 \Rightarrow u=-1$
$x=1 \Rightarrow u=0$

$$= \frac{2}{3}\sin(u)\Big|_{-1}^{0} = \frac{2}{3}\sin(0) - \frac{2}{3}\sin(-1) = \boxed{-\frac{2}{3}\sin(-1)}$$

$$\hookrightarrow = \boxed{\frac{2}{3}\sin(1)}$$

5. (5 pts.) Sketch (and label axes, boundaries of solid, etc. appropriately) the solid whose volume is computed by the interated integral given in polar coordinates below:

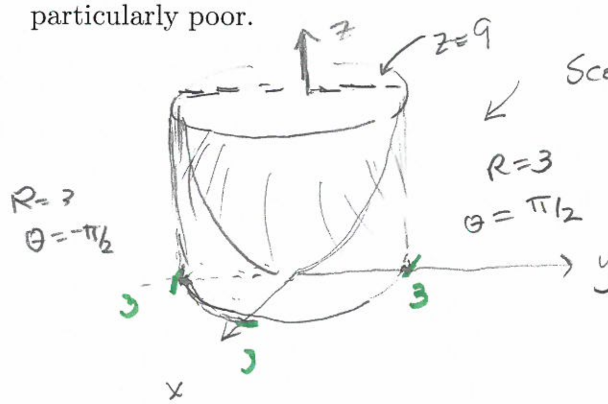
$$\int_0^3 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r^3\,d\theta\,dr$$

Hint: converting the integrand to rectangular coordinates might help, if you are stuck. Second hint: describe the solid in words if your sketch is particularly poor.

$$= \int_0^3 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \underbrace{r^2}_{x^2+y^2} \underbrace{(r\,dr\,d\theta)}_{dA}$$

$0 \le r \le 3$

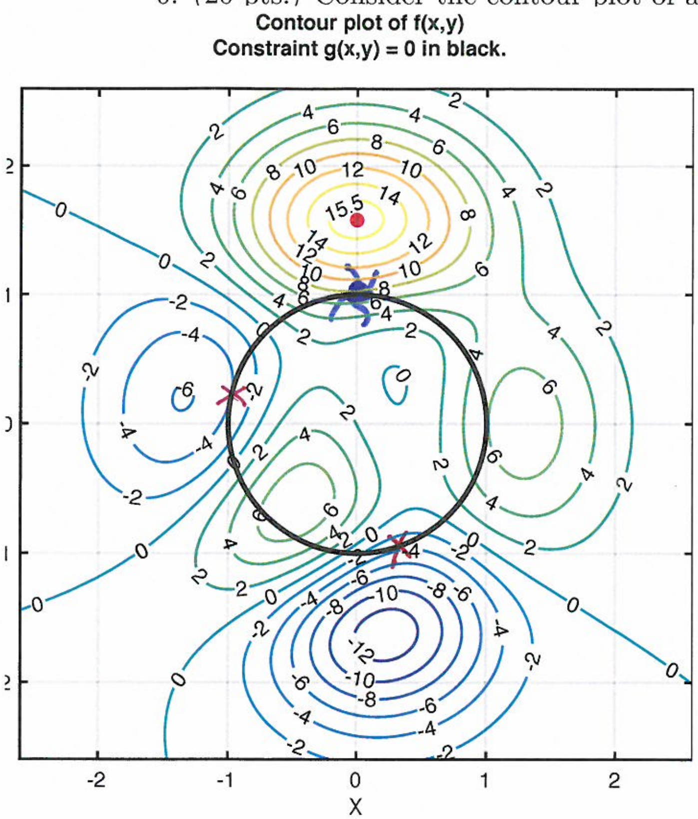
$-\pi/2 \le \theta \le \pi/2$



Solid obtained by Scooping a "bowl" out from the half-cylinder.

Solid below $z = x^2+y^2$

6. (20 pts.) Consider the contour plot of a continuous function $f(x, y)$ below.



(a) (12 pts. – 4 pts each)

The point $P(0, 1.58)$ is marked with a red dot in the plot.

i. What is the value of $f_x(0, 1.58)$? Why?

$\boxed{0}$ (0, 1.58) is a local maximum

⇒ critical point

ii. What is the value of $f_{yy}(0, 1.58)$? Why?

negative local max!

this indicates f(x, y) is

concave down.

iii. Give the equation of the tangent plane at $(0, 1.58, f(0, 1.58))$.

Answer: $z = 16$ (or 15.5 or something similar

(b) (8pts.) The black constraint curve has equation $g(x, y) = x^2 + y^2 - 1 = 0$.

i. (5 pts.) By inspection, estimate the maximum and minimum values of $f(x, y)$ subject to the constraint $x^2 + y^2 = 1$ and find the points (a, b) where they occur.

Answer:

The minimum value is approximately -4 and occurs at the point(s) $(-1, 0.3)$ $(0.3, -1)$. marked with an x

The maximum value is approximately 8 and occurs at the point(s) $(0, 1)$. marked with

ii. (3 pts. — no partial credit) According to the method of Lagrange multipliers, at the points (a, b) where the minimum and maximum values occur, the equation

$$\nabla g(a, b) = \lambda \nabla f(a, b)$$

holds. Explain what this means and how this relates to the contours of $f(x, y)$ and the constraint equation $g(x, y) = 0$.

The gradient vectors are parallel and orthogonal to level curves which are tangent at max/min. values

7. (5 pts.) Suppose that $g(x, y, z)$ is a function of three variables, where the coordinate functions are given by $x(u, v)$, $y(u, v)$, and $z(u, v)$.

Use the Chain Rule to give a formula for the partial derivative $\dfrac{\partial g}{\partial v}$.

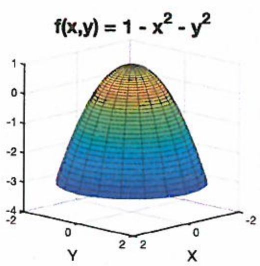
$$\frac{\partial g}{\partial v} = \frac{\partial g}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial g}{\partial z}\frac{\partial z}{\partial v}$$

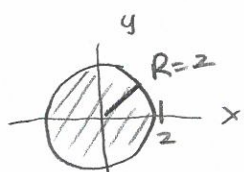
8. (10 pts.) Graphed below is the part of the paraboloid $z = 1 - x^2 - y^2$ that lies above the plane $z = -3$. Set up **in polar coordinates**, but **do not evaluate**, an iterated integral that calculates the surface area of this surface. A complete and correct answer is in terms of r, θ only, (no x, no y), has correct limits of integration and the integrand simplified.

$-3 = 1 - x^2 - y^2$

$\Rightarrow x^2 + y^2 = 4$

$R^2 = 4$





Domain of Integration

$0 \leq r \leq 2$

$0 \leq \theta \leq 2\pi$

$$\iint_R \sqrt{1 + f_x^2 + f_y^2}\, dA$$

$$= \iint_R \sqrt{1 + (-2x)^2 + (-2y)^2}\, dA = \iint_R \sqrt{1 + 4(x^2+y^2)}\, dA = \boxed{\int_0^{2\pi}\int_0^2 \sqrt{1+r^2}\; r\,dr\,d\theta}$$

Extra Credit: $SA = \int_0^{2\pi}\int_0^2 r(1+4r^2)^{\frac{1}{2}}\,dr\,d\theta = 2\pi \int_0^2 r(1+4r^2)^{\frac{1}{2}}\,dr$ $\quad u = 1+4r^2$, $du = 8r\,dr$ ...

$$= 2\pi\left[\frac{1}{12}(1+4r^2)^{\frac{3}{2}}\right]_0^2 = \frac{2\pi}{12}\left[17^{3/2} - 1\right] = \frac{\pi}{6}(17\sqrt{17} - 1) \text{ or }$$

$$\frac{\pi}{6}(17^{3/2} - 1)$$

Extra credit (5 pts.): On a scratch piece of paper, compute the surface area. (This is totally doable.)